PROOF OF TWO-DIMENSIONAL JACOBIAN CONJECTURE

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Abstract. Let $(F,G)\in\mathbb{C}[x,y]^2$ be a Jacobian pair and $\sigma:(a,b)\mapsto(F(a,b),G(a,b))$ for $(a,b)\in\mathbb{C}^2$ the corresponding Keller map. The local bijectivity of Keller maps tells that for $p\in\mathbb{C}^2$, there exist neighborhoods $\mathcal{O}_p$ of p and $\mathcal{O}_{\sigma(p)}$ of $\sigma(p)$ such that $\sigma_p=\sigma|_{\mathcal{O}_p}:\mathcal{O}_p\to\mathcal{O}_{\sigma(p)}$ is a bijection. Thus if there exist $p_0,p_1\in\mathbb{C}^2$ with $p_0\ne p_1$, $\sigma(p_0)=\sigma(p_1)$, then the local bijectivity implies that $\sigma_{p_1}^{-1}\sigma_{p_0}:\mathcal{O}_{p_0}\to\mathcal{O}_{p_1}$ is a bijection between some neighborhoods of p_0 and p_1. We generalize this result in various aspects, which lead us to give a proof of injectivity of Keller maps and thus the 2-dimensional Jacobian conjecture. Among those generalizations, one is the following (cf. Theorem 1.5): For any $(p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)\in\mathbb{C}^2\times\mathbb{C}^2$ satisfying $p_0\ne p_1$, $\sigma(p_0)=\sigma(p_1)$, $\kappa_0\le\kappa_1|x_1|^{\kappa_2}\le\kappa_3|x_0|\le\kappa_4|x_1|^{\kappa_5}\le\kappa_6$, $\ell_{p_0,p_1}:=\frac{|y_1|+\kappa_7}{|x_1|^{\kappa_8}}\ge\kappa_9$ for some preassigned $\kappa_i\in\mathbb{R}_{>0}$, there exists $(q_0,q_1)\in\mathbb{C}^2\times\mathbb{C}^2$ satisfying the same conditions, and furthermore $\ell_{q_0,q_1}>\ell_{p_0,p_1}$.

Contents

1. Main theorem

Let us start with an arbitrary *Jacobian pair* $(F,G)\in\mathbb{C}[x,y]^2$, i.e., a pair of polynomials on two variables x,y with a nonzero constant *Jacobian determinant*

$$J(F,G):=\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}\end{vmatrix}\in\mathbb{C}_{\ne0}. \tag{1.1}$$

Assume that the corresponding *Keller map* $\sigma:\mathbb{C}^2\to\mathbb{C}^2$ sending, for $p=(a,b)\in\mathbb{C}^2$,

$$p\mapsto\big(F(p),G(p)\big):=\big(F(a,b),G(a,b)\big), \tag{1.2}$$

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is not injective, namely, for some $p_0 = (x_0, y_0),\ p_1 = (x_1, y_1) \in \mathbb{C}^2$,

$$\sigma(p_0) = \sigma(p_1), \quad p_0 \neq p_1. \tag{1.3}$$

The local bijectivity of Keller maps says that for any $p \in \mathbb{C}^2$, there exist neighborhoods $\mathcal{O}_p$ of p and $\mathcal{O}_{\sigma(p)}$ of $\sigma(p)$ such that $\sigma_p = \sigma|_{\mathcal{O}_p}$ is a bijection between these two neighborhoods. This implies that $\sigma_{p_1}^{-1}\sigma_{p_0} : \mathcal{O}_{p_0} \to \mathcal{O}_{p_1}$ is a bijection between some neighborhoods $\mathcal{O}_{p_0}$ of p_0 and $\mathcal{O}_{p_1}$ of p_1 (we may assume $\mathcal{O}_{p_0}$ and $\mathcal{O}_{p_1}$ are disjoint), i.e., any $q_0 \in \mathcal{O}_{p_0}$ is in 1–1 correspondence with $q_1 \in \mathcal{O}_{p_1}$ such that $\sigma(q_0) = \sigma(q_1)$ and $q_0 \neq q_1$. In this paper we generalize this result in various aspects, which lead us to present a proof of injectivity of Keller maps, which implies the well-known Jacobian conjecture (see, e.g., the References).

Theorem 1.1. (Main Theorem) *Let $(F, G) \in \mathbb{C}[x, y]^2$ be a Jacobian pair. Then the Keller map σ is injective. In particular, the 2-dimensional Jacobian conjecture holds, i.e., F, G are generators of $\mathbb{C}[x, y]$.*

First we give some formulations. Fix (once and for all) a sufficiently large $\ell \in \mathbb{Z}_{>0}$. Applying the following variable change,

$$(x, y) \mapsto \big(x + (x + y)^{\ell},\ x + y\big), \tag{1.4}$$

and rescaling F, G, we can assume, for some $m \in \mathbb{Z}_{>0}$,

$$\operatorname{Supp} F \subset \Delta_{0,\xi,\eta}, \qquad F_L = (x + y)^m, \qquad J(F, G) = 1, \tag{1.5}$$

where

- $\operatorname{Supp} F := \big\{(i, j) \in \mathbb{Z}^2_{\geq 0} \,|\, \mathrm{C_{oeff}}(F, x^i y^j) \neq 0\big\}$ is the *support* of F [cf. Convention 2.1 (2) (iv) for notation $\mathrm{C_{oeff}}(F, x^i y^j)$],
- $\Delta_{0,\xi,\eta}$ is the triangular with vertices $0 = (0, 0)$, $\xi = (m, 0)$, $\eta = (0, m)$,
- L is the edge of $\operatorname{Supp} F$ linking vertices ξ, η,
- F_L, which we refer to as the *leading part* of F, is the part of F corresponding to the edge L [which means that $\operatorname{Supp} F_L = L \cap \operatorname{Supp} F = \{(m - i, i) \in \mathbb{C}^2 \,|\, i = 0, 1, ..., m\}$].

The reason we take the variable change (1.4) is to use the leading part F_L of F to control F in some sense [cf. (3.11)], which guides us to obtain Theorem 1.3.

Throughout the paper, we use the following notations,

$$(p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big) \in \mathbb{C}^2 \times \mathbb{C}^2 \cong \mathbb{C}^4, \tag{1.6}$$

$$V = \big\{(p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big) \in \mathbb{C}^4 \,\big|\, \sigma(p_0) = \sigma(p_1),\ p_0 \neq p_1\big\}, \tag{1.7}$$

$$V_{\xi_0,\xi_1} = \big\{(p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big) \in V \,\big|\, x_0 = \xi_0,\ x_1 = \xi_1\big\}, \tag{1.8}$$

for any $\xi_0, \xi_1 \in \mathbb{C}$. Then $V \neq \emptyset$ by assumption (1.3). The main result used in the proof of Theorem 1.1 is the following.

Theorem 1.2. (i) *There exist $\xi_0, \xi_1 \in \mathbb{C}$ such that $V_{\xi_0,\xi_1} = \emptyset$.*

(ii) *Fix any $\xi_0, \xi_1 \in \mathbb{C}$ satisfying* (i)*. Denote, for $(p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big) \in V$,*

$$\boldsymbol{d}_{p_0,p_1} = |x_0 - \xi_0|^2 + |x_1 - \xi_1|^2. \tag{1.9}$$

Then for any $(p_0,p_1)\in V$, *there exists* $(q_0,q_1)=\big((\dot x_0,\dot y_0),(\dot x_1,\dot y_1)\big)\in V$ *such that*

$$\boldsymbol{d}_{q_0,q_1}<\boldsymbol{d}_{p_0,p_1}. \tag{1.10}$$

After a proof of this result, it is then not surprising that it can be used to give a proof of Theorem 1.1 by taking some kind of "limit" [cf. (7.2)], which can guide us to derive a contradiction. We would like to mention that at a first sight, Theorem 1.2 (i) seems to be obvious, however its proof is highly nontrivial to us, it needs several results, which we state below. Here is the first one.

Theorem 1.3. *Denote, for* $(p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)\in V$,

$$h_{p_0,p_1}=\max\big\{|x_1|,|y_1|,|x_0|,|y_0|\big\}, \tag{1.11}$$

and call h_{p_0,p_1} *the* **height** *of* (p_0,p_1). *There exists* $\boldsymbol{s}_0\in\mathbb{R}_{>0}$ (*depending on* $m=\deg F$, $\deg G$ *and coefficients of* F *and* G*; cf. Remark* 3.5) *satisfying the following: For any* $(p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)\in V$ *with*

$$h_{p_0,p_1}\geq\boldsymbol{s}_0, \tag{1.12}$$

we must have

$$|x_0+y_0|<h_{p_0,p_1}^{\frac{m}{m+1}},\qquad |x_1+y_1|<h_{p_0,p_1}^{\frac{m}{m+1}}. \tag{1.13}$$

In particular if $h_{p_0,p_1}=\max\big\{|x_t|,|y_t|\big\}$ *for some* $t\in\{0,1\}$, *then,*

$$|a-b|<n_t^{\frac{m+1}{m+2}}, \tag{1.14}$$

for any $a,b\in\big\{|x_t|,|y_t|,h_{p_0,p_1}\big\}$, *where* $n_t=\min\big\{|x_t|,|y_t|\big\}$.

To prove Theorem 1.2 (i), we assume conversely that, for all $\xi_0,\xi_1\in\mathbb{C}$,

$$V_{\xi_0,\xi_1}\neq\emptyset. \tag{1.15}$$

Then we are able to obtain the following.

Theorem 1.4. *Under the assumption* (1.15), *we have the following.*

(i) *The following subset of* V *is a nonempty closed bounded subset of* $\mathbb{C}^4$ *for any* $k_0,k_1\in\mathbb{R}_{\geq0}$,

$$A_{k_0,k_1}=\big\{(p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)\in V\ \big|\ |x_0|=k_0,\ |x_1|=k_1\big\}. \tag{1.16}$$

(ii) *The following is a well-defined function on* $k_0,k_1\in\mathbb{R}_{\geq0}$,

$$\gamma_{k_0,k_1}=\max\big\{|y_1|\ \big|\ (p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)\in A_{k_0,k_1}\big\}. \tag{1.17}$$

(iii) *The* γ_{k_0,k_1} *is an "almost strictly" increasing function on both variables* $k_0,k_1\in\mathbb{R}_{\geq0}$ *in the following sense* [*we need to require* $k_0>0$ *in* (1.18) (b), *that is why we use the words "almost strictly"*],

$$\begin{aligned}&\text{(a)}\ \ \gamma_{k_0',k_1}>\gamma_{k_0,k_1}\quad \text{if}\ \ k_0'>k_0\geq0,\ k_1\geq0,\\ &\text{(b)}\ \ \gamma_{k_0,k_1'}>\gamma_{k_0,k_1}\quad \text{if}\ \ k_0>0,\ k_1'>k_1\geq0.\end{aligned} \tag{1.18}$$

This result is then used to prove the following (which is the hardest part of the paper).

Theorem 1.5. (1) *There exist* $\kappa_i\in\mathbb{R}_{>0}$, *such that the following hold.*

(i) *Denote by* V_0 *the subset of* V *such that all its elements* $(p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)$ *simultaneously satisfy one* (*and the only one*) *of* (1.19) *or* (1.20). *Then* $V_0\neq\emptyset$.

$$\text{(a) } \kappa_0\le\kappa_1|x_1|^{\kappa_2}\le\kappa_3|x_0|\le\kappa_4|x_1|^{\kappa_5}\le\kappa_6,\quad \text{(b) } \ell_{p_0,p_1}:=\frac{|y_1|+\kappa_7}{|x_1|^{\kappa_8}}\ge\kappa_9; \tag{1.19}$$

$$\begin{aligned}&\text{(a) } \kappa_0\le|f_1|^{\kappa_1}\le|f_2|\le|f_3|^{\kappa_2}\le\kappa_3, \qquad \text{(b) } \kappa_4|f_1|^{-\kappa_5}\le|x_0|\le\kappa_6,\\ &\text{(c) } \kappa_7|f_1|^{\kappa_8}\le|x_1|\le\kappa_9,\quad \text{(d) } \ell_{p_0,p_1}:=|f_4f_1^{\kappa_{10}}|+|x_1|+|y_1|\ge\kappa_{11},\end{aligned} \tag{1.20}$$

where, f_i*'s are some locally smooth functions on* x_0,x_1,y_1*, and* κ_i *will be chosen such that there exist* $\theta_i\in\mathbb{R}_{>0}$ *satisfying: when conditions hold, we have*

$$\theta_0\le|x_0|,|x_1|\le\theta_1,\qquad |y_1|\ge\theta_0. \tag{1.21}$$

(ii) *For any* $(p_0,p_1)\in V_0$*, no equality can occur in the first or last inequality of* (1.19) (a) *or* (1.20) (a)*, or in the any inequality of* (1.20) (b) *or* (c) *; further, two equalities cannot simultaneously occur in the second and third inequalities of* (1.19) (a) *or* (1.20) (a).

(2) *For any* $(p_0,p_1)\in V_0$*, there exists* $(q_0,q_1)=\big((\dot x_0,\dot y_0),(\dot x_1,\dot y_1)\big)\in V_0$ *such that*

$$\ell_{q_0,q_1}>\ell_{p_0,p_1}. \tag{1.22}$$

Remark 1.6. (1) Throughout the paper, we will frequently use the local bijectivity of Keller maps. Theorem 1.5 (2) says that [assume for example, we have case (1.19)]

$$\begin{aligned}&\text{(a) } \kappa_0\le\kappa_1|\dot x_1|^{\kappa_2}\le\kappa_3|\dot x_0|\le\kappa_4|\dot x_1|^{\kappa_5}\le\kappa_6, &&\text{(b) } \frac{|\dot y_1|+\kappa_7}{|\dot x_1|^{\kappa_8}}>\frac{|y_1|+\kappa_7}{|x_1|^{\kappa_8}},\\ &\text{(c) } \sigma(q_0)=\sigma(q_1), &&\text{(d) } q_0\neq q_1.\end{aligned} \tag{1.23}$$

If we regard $\dot x_0,\dot y_0,\dot x_1,\dot y_1$ as 4 free variables, then the local bijectivity always allows us to obtain (1.23) (c), which imposes two restrictions on 4 variables. We can impose at most two more "nontrivial" restrictions on them [we regard (1.23) (d) as a trivial restriction, see below]. The main difficulty for us is how to impose two more "solvable" restrictions on variables [see (3) below] to control $\dot x_0,\dot y_0,\dot x_1,\dot y_1$ in order to achieve our goal of "deriving a contradiction". However, it seems to us that two more restrictions are always insufficient to achieve the goal. Here, condition (1.23) (b) imposes one more restriction, and we have one free variable left. However there are 4 restrictions in (1.23) (a), thus in general there will be no solutions. Thanks to Theorem 1.5 (1) (ii) [see (2) below], we only need to take care of one restriction in (1.23) (a) each time [cf. (6.6)] since we are always under a "local" situation (i.e., we are only concerned with a small neighborhood of some points each time), and thus the inequation in (1.23) (a) is solvable [we do not need to consider condition (1.23) (d) under the "local" situation, we only need to take care of it when we take some kind of "limit", cf. (7.2)].

(2) Condition (1.23) (b) is not only used to control $|\dot x_1|$ and $|\dot y_1|$, but also used to take the "limit"; while (1.23) (a) is used to control $|\dot x_0|$ and $|\dot x_1|$. We remark that the requirement "$\kappa_7>0$" in (1.19) (b) [or the last two terms in (1.20) (c)] will guarantee that the correspondent inequation (1.23) (b) is solvable [see (3) below, cf. Remark 6.2]. Finally we would like to mention that to find conditions like the ones in (1.19) or (1.20) satisfying Theorem

1.5 (1) (ii) has been extremely difficult for us, we achieve this by using Theorem 1.4 to prove several technical lemmas (cf. Assumption 5.1 and Lemmas 5.2–5.11).

(3) One may expect to have some statements such as one of the following:

(i) For any $(p_0, p_1) \in V$ with $|x_0|^{\theta_0} + |x_1|^{\theta_1} \le \boldsymbol{s}$ (for some $\theta_i, \boldsymbol{s} \in \mathbb{R}_{>0}$), there exists $(q_0, q_1) \in V$ such that

$$\text{(a) } |\dot{x}_0|^{\theta_0} + |\dot{x}_1|^{\theta_1} \le \boldsymbol{s}, \qquad \text{(b) } |\dot{y}_1|^{\theta_2} > |y_1|^{\theta_2}. \tag{1.24}$$

(ii) For any $(p_0, p_1) \in V$ with $|x_1|^{\theta_1} \le |y_1|^{\theta_2} + \boldsymbol{s}$, there exists $(q_0, q_1) \in V$ such that

$$\text{(a) } |\dot{x}_1|^{\theta_1} \le |\dot{y}_1|^{\theta_2} + \boldsymbol{s}, \qquad \text{(b) } |\dot{y}_0|^{\theta_3} - |\dot{x}_0|^{\theta_4} > |y_0|^{\theta_3} - |x_0|^{\theta_4}. \tag{1.25}$$

If a statement such as one of the above could be obtained, then a proof of Theorem 1.1 would be easier. At a first sight, the condition (1.24) (a) [or (1.10)] only imposes one restriction on variables, however it in fact contains 2 hidden restrictions [see arguments after (7.7)] simply because the left-hand side of "$\le$" has 2 positive terms with absolute values containing variables. The second condition in (1.25) is unsolvable as will be explained in Remark 6.2. We would also like to point out that to obtain Theorem 1.1, we always need to take some kind of "limit" [cf. (7.2)] to derive a contradiction. Thus some condition such as (1.10), (1.23) (b), (1.24) (b) or (1.25) (b) is necessary in order to take the "limit".

2. Some preparations

We need some conventions and notations, which, for easy reference, are listed as follows.

Convention 2.1. (1) A complex number is written as $a = a_{\rm re} + a_{\rm im}\boldsymbol{i}$ for some $a_{\rm re}, a_{\rm im} \in \mathbb{R}$, where $\boldsymbol{i} = \sqrt{-1}$. If a^b appears in an expression, then we always assume $b \in \mathbb{R}$, and in case $a \ne 0$, we interpret a^b as the unique complex number $r^b e^{b\theta\boldsymbol{i}}$ by writing $a = re^{\theta\boldsymbol{i}}$ for some $r \in \mathbb{R}_{>0}$, $-\pi < \theta \le \pi$ and e is the natural number.

(2) Let $P = \sum_{i\in\mathbb{Z}_{\ge 0}} p_i y^{\alpha+i} \in \mathbb{C}(x)((y))$ with $\alpha \in \mathbb{Z}$, $p_i \in \mathbb{C}(x)$.

(i) Assume $p_0 = 1$. For any $\beta \in \mathbb{Q}$ with $\alpha\beta \in \mathbb{Z}$, we always interpret P^β as

$$P^\beta = y^{\alpha\beta}\Big(1 + \sum_{j=1}^{\infty} \binom{\beta}{j}\Big(\sum_{i\in\mathbb{Z}_{>0}} p_i y^i\Big)^j\Big) \in \mathbb{C}(x)((y)), \tag{2.1}$$

where in general, we denote the *multi-nomial coefficient*

$$\binom{k}{\lambda_1, \lambda_2, ..., \lambda_i} = \frac{k(k-1)\cdots(k-(\lambda_1+\lambda_2+\cdots+\lambda_i)+1)}{\lambda_1!\lambda_2!\cdots\lambda_i!}. \tag{2.2}$$

Then

$$(P^\beta)^{\beta'} = P^{\beta\beta'}, \tag{2.3}$$

for any $\beta \in \mathbb{Q}$, $\beta' \in \mathbb{Z}$ with $\alpha\beta, \alpha\beta\beta' \in \mathbb{Z}$. If $p_0 \ne 1$, then p_0^β is in general a multi-valued function on p_0, and if we fix a choice of p_0^β, then (2.3) only holds when $\beta' \in \mathbb{Z}$ [fortunately we will only encounter this situation, cf. (3.23) and statements after it].

(ii) For $Q_1, Q_2 \in \mathbb{C}(x)((y))$, we use the following notation [as long as it is algebraically a well-defined element in $\mathbb{C}(x)((y))$]

$$P(Q_1, Q_2) = P|_{(x,y)=(Q_1,Q_2)} = \sum_i p_i(Q_1) Q_2^{\alpha+i}. \tag{2.4}$$

(iii) If $Q_1, Q_2 \in \mathbb{C}$, and $Q_2 \neq 0$ in case (2.4) contains some negative powers of Q_2, we also use (2.4) to denote a well-defined complex number as long as $p_i(Q_1)$ exists for all posible i and the series (2.4) converges absolutely.

(iv) For any $Q = \sum_{i\in\mathbb{Z}_{\ge 0}} q_i y^{\beta+i} \in \mathbb{C}(x)((y))$, by comparing coefficients of $y^{\beta+i}$ for $i \ge 0$, there exists uniquely $b_i \in \mathbb{C}(x)$ such that

$$Q = \sum_{i=0}^{\infty} b_i P^{\frac{\beta+i}{\alpha}}. \tag{2.5}$$

We call b_i the *coefficient of* $P^{\frac{\beta+i}{\alpha}}$ *in* Q, and denote by $\mathrm{C_{oeff}}(Q, P^{\frac{\beta+i}{\alpha}})$. If $Q = \sum_{i,j} q_{ij} x^i y^j$ with $q_{ij} \in \mathbb{C}$, we also denote $\mathrm{C_{oeff}}(Q, x^i y^j) = q_{ij}$.

(3) Throughout the paper, we need two (and sometimes more) independent parameters $\boldsymbol{k} \gg 1$ (i.e., $\boldsymbol{k} \to \infty$) and $\mathcal{E} \to 0$. We use the following convention: Symbols $\boldsymbol{s}, \boldsymbol{s}_j$ for $j \ge 0$ always denote some (possibly sufficiently large) numbers independent of $\mathcal{E}, \boldsymbol{k}$. We use $O(\mathcal{E})^i$ for $i \in \mathbb{Q}_{\ge 0}$ to denote any element P in $\mathbb{C}(x)((y))$ (or especially in $\mathbb{C}$) such that $P(\dot{x}, \dot{y})$ converges absolutely and $|\mathcal{E}^{-i} P(\dot{x}, \dot{y})| < \boldsymbol{s}$ for some fixed $\boldsymbol{s}$, where $(\dot{x}, \dot{y})$ is in some required region which will be specified in the context.

Let $P = \sum_j p_j y^j \in \mathbb{C}(x)((y))$, $p_j \in \mathbb{C}(x)$, and $(x_0, y_0) \in \mathbb{C}^2$ with $y_0 \neq 0$. If $p_j(x_0)$ exists for all possible j, and $z_0 = \sum_j |p_j(x_0) y_0^j|$ converges, then z_0 is called the *absolute converging value* of P at (x_0, y_0), denoted by $A_{(x_0,y_0)}(P)$ [or by $A_{(y_0)}(P)$ if P does not depend on x].

Definition 2.2. (1) Let P be as above and $Q = \sum_i q_i y^i \in \mathbb{C}((y))$, $q_i \in \mathbb{R}_{\ge 0}$, $x_0 \in \mathbb{C}$. If $p_i(x_0)$ exists and

$$|p_i(x_0)| \le q_i, \tag{2.6}$$

for all possible i, then we say Q is a *controlling function* for P on y at point x_0, and denote

$$P \trianglelefteq_y^{x_0} Q \quad \text{or} \quad Q \trianglerighteq_y^{x_0} P, \tag{2.7}$$

or $P \trianglelefteq_y Q$ or $Q \trianglerighteq_y P$ when there is no confusion. In particular if P, Q do not depend on y then we write $P \trianglelefteq^{x_0} Q$ or $Q \trianglerighteq^{x_0} P$ (thus $a \trianglelefteq b$ for $a, b \in \mathbb{C}$ simply means that $|a| \le b$ with $b \ge 0$).

(2) An element in $\mathbb{C}((y))$ with non-negative coefficients (such as Q above) is called a *controlling function* on y.

(3) If $Q = q_0 y^\alpha + \sum_{j>0} q_j y^{\alpha+j} \in \mathbb{C}((y))$ is a controlling function on y with $q_i \in \mathbb{R}_{\ge 0}$ and $q_0 > 0$, then we always use the same symbol with subscripts "$_{\rm igo}$" and "$_{\rm neg}$" to denote the elements

$$Q_{\rm igo} = q_0^{-1} \sum_{j>0} q_j y^j, \quad Q_{\rm neg} := q_0 y^\alpha \Big(1 - q_0^{-1} \sum_{j>0} q_j y^j\Big) = q_0 y^\alpha (1 - Q_{\rm igo}) = 2 q_0 y^\alpha - Q. \tag{2.8}$$

We call $Q_{\rm igo}$ the *ignored part* of Q, and $Q_{\rm neg}$ the *negative correspondence of* Q [in sense of (2.10) and (2.11), where $a, -k$ are nonpositive].

Lemma 2.3. (1) *If*

$$P = p_0 y^\alpha + \sum_{j>0} p_j y^{\alpha+j} \in \mathbb{C}(x)((y)), \quad Q = q_0 y^\alpha + \sum_{j>0} q_j y^{\alpha+j} \in \mathbb{C}((y)), \tag{2.9}$$

with $P \trianglelefteq_y^{x_0} Q$, $x_0 \in \mathbb{C}$ *and* $|p_0(x_0)| = q_0 \in \mathbb{R}_{>0}$, *then*

$$\text{(a)}\ \frac{\partial P}{\partial y} \ \trianglelefteq_y^{x_0}\ \pm\frac{dQ}{dy}, \qquad \text{(b)}\ P^a \ \trianglelefteq_y^{x_0}\ Q_{\rm neg}^a \ \trianglelefteq_y\ (q_0 y^\alpha)^{-b} Q_{\rm neg}^{a+b} \text{ for } a,b \in \mathbb{Q}_-, \tag{2.10}$$

$$Q^k \ \trianglelefteq_y\ (q_0y^\alpha)^{2k}Q_{\rm neg}^{-k} \ \trianglelefteq_y \begin{cases} \dfrac{(q_0y^\alpha)^k}{1-kQ_{\rm igo}} & \text{if } k\in\mathbb{Z}_{\geq 1},\\ (q_0y^\alpha)^k\Big(1+\dfrac{kQ_{\rm igo}}{1-Q_{\rm igo}}\Big) & \text{if } k\in\mathbb{Q}_{\geq 0} \text{ with } k<1. \end{cases} \tag{2.11}$$

where (2.10) (*a*) *holds under the condition: either both* P *and* Q *are power series of* y (*in this case the sign is "*$+$*"*), *or else both are polynomials on* y^{-1} (*in this case the sign is "*$-$*"*).

(2) *If* $x_0, y_0 \in \mathbb{C}$ *with* $y_0 \neq 0$, *and* $P_1 \trianglelefteq_y^{x_0} Q_1$, $P_2 \trianglelefteq_y^{x_0} Q_2$, *then*

$$A_{(x_0,y_0)}(P_1P_2) \leq A_{(y_0)}(Q_1)A_{(y_0)}(Q_2) = Q_1(|y_0|)Q_2(|y_0|). \tag{2.12}$$

Proof. One can see that (2) and (2.10) (a) are obvious, and (2.10) (b), (2.11) are obtained by noting that for $a, b \in \mathbb{Q}_-$ and $i \in \mathbb{Z}_{>0}$, one has

$$(-1)^i\binom{a}{i} = \Big|\binom{a}{i}\Big| \leq \Big|\binom{a+b}{i}\Big| = (-1)^i\binom{a+b}{i}, \quad \binom{k}{i} \leq \Big|\binom{-k}{i}\Big| \leq \begin{cases} k^i & \text{if } k\in\mathbb{Z}_{\geq 1},\\ k & \text{if } 0<k\in\mathbb{Q}_{<1}.\end{cases}$$

This proves the lemma. □

Take

$$\tilde F = \tilde f_1 y + \sum_{i=2}^{\infty} \tilde f_i y^i \in \mathbb{C}[x][[y]], \tag{2.13}$$

with $\tilde f_i \in \mathbb{C}[x]$ and $\tilde f_1 \neq 0$. Regarding $\tilde F$ as a formal function on y (with parameter x being regarded as fixed), we have the *formal inverse function* denoted by $y_{\tilde F} \in \mathbb{C}[x, \tilde f_1^{-1}][[\tilde F]] \subset \mathbb{C}(x)[[\tilde F]]$ such that [cf. (2.5)]

$$y \ = \ y_{\tilde F}(\tilde F) = \boldsymbol{b}_1 \tilde F + \sum_{i=2}^{\infty} \boldsymbol{b}_i \tilde F^i, \tag{2.14}$$

with $\boldsymbol{b}_i = \mathrm{C}_{\rm oeff}(y, \tilde F^i) \in \mathbb{C}[x, \tilde f_1^{-1}]$ being determined by $\boldsymbol{b}_1 = \tilde f_1^{-1} \in \mathbb{C}[x, \tilde f_1^{-1}]$ and (we do not need to use the following explicit expression of $\boldsymbol{b}_i$, we only want to present that $\boldsymbol{b}_i$'s exist)

$$\boldsymbol{b}_i \ = \ -\sum_{j=1}^{i-1} \boldsymbol{b}_j \tilde f_1^{j-i} \sum_{\ell=0}^{j}\binom{j}{\ell} \sum_{\substack{n\in\mathbb{Z}_{\geq 0},\,\lambda_1,\lambda_2,\dots,\lambda_n\geq 0\\ \lambda_1+2\lambda_2+\cdots+n\lambda_n=i-j}} \binom{\ell}{\lambda_1,\lambda_2,\dots,\lambda_n} \tilde f_1^{-\lambda_1-\lambda_2-\cdots-\lambda_n} \tilde f_2^{\lambda_2} \tilde f_3^{\lambda_3} \cdots \tilde f_n^{\lambda_n}, \tag{2.15}$$

for $i \geq 2$, which is obtained by comparing the coefficients of y^i in (2.14).

Lemma 2.4. *For* $\hat a_i \in \mathbb{R}_{\geq 0}$ *with* $\hat a_1 > 0$, *let*

$$\hat{\boldsymbol{F}} = \hat a_1 y + \sum_{i=2}^{\infty} \hat a_i y^i \in \mathbb{C}[[y]] \quad \text{and} \quad \hat{\boldsymbol{F}}_{\rm neg} = \hat a_1 y - \sum_{i=2}^{\infty} \hat a_i y^i, \tag{2.16}$$

be a controlling function on y and its negative correspondence [cf. (2.8)], and let

$$y = y^{\rm neg}(\hat{\boldsymbol F}_{\rm neg}) = \hat{\boldsymbol b}_1 \hat{\boldsymbol F}_{\rm neg} + \sum_{i=2}^{\infty} \hat{\boldsymbol b}_i \hat{\boldsymbol F}^i_{\rm neg}, \tag{2.17}$$

be the formal inverse function of $\hat{\boldsymbol F}_{\rm neg}$, where $\hat{\boldsymbol b}_1 = \hat{a}_1^{-1}$ and $\hat{\boldsymbol b}_i = {\rm C_{oeff}}(y, \hat{\boldsymbol F}^i_{\rm neg}) \in \mathbb{C}$. Then

(1) *$y^{\rm neg}(\hat{\boldsymbol F}_{\rm neg})$ is a controlling function on $\hat{\boldsymbol F}_{\rm neg}$, i.e., for $i \ge 1$,*

$$\hat{\boldsymbol b}_i = {\rm C_{oeff}}(y, \hat{\boldsymbol F}^i_{\rm neg}) \ge 0. \tag{2.18}$$

(2) *If $\tilde{F} \trianglelefteq^{x_0}_y \hat{\boldsymbol F}$ with $\tilde{F}$ as in (2.13) and $\tilde{f}_i(x_0)$ exists for all possible i and $|\tilde{f}_1(x_0)| = \hat{a}_1$, then*

$$y = y_{\tilde F}(\tilde{F}) \trianglelefteq^{x_0}_{\tilde F} y^{\rm neg}(\tilde{F}), \quad \textit{i.e.,} \quad \boldsymbol{b}_i \trianglelefteq^{x_0} \hat{\boldsymbol b}_i, \tag{2.19}$$

where $\boldsymbol{b}_i = {\rm C_{oeff}}(y, \tilde{F}^i)$ is as in (2.14), and $\boldsymbol{b}_i \trianglelefteq^{x_0} \hat{\boldsymbol b}_i$ means that $|\boldsymbol{b}_i(x_0)| \le \hat{\boldsymbol b}_i$. In particular

$$y \trianglelefteq_y y^{\rm neg}(\hat{\boldsymbol F}), \tag{2.20}$$

where the right side of "$\trianglelefteq_y$" is regarded as a function on y by substituting $\hat{\boldsymbol F}$ by (2.16).

Proof. Note that (1) follows from (2) by simply taking $\tilde{F} = \hat{a}_1 y$. Thus we prove (2). We want to prove, for $i \ge 1$,

$$\frac{\partial^i y}{\partial \tilde{F}^i} \trianglelefteq^{x_0}_y \frac{d^i y}{d\hat{\boldsymbol F}^i_{\rm neg}}, \tag{2.21}$$

where the left-hand side is understood as that we first use (2.14) to regard y as a function on $\tilde{F}$ (with parameter x) and apply $\frac{\partial^i}{\partial \tilde{F}^i}$ to it, then regard the result as a function on y (and the like for the right-hand side, which does not contain the parameter x). By (2.10) (a), we have $\frac{\partial \tilde F}{\partial y} \trianglelefteq^{x_0}_y \frac{d\hat{\boldsymbol F}}{dy}$, and thus by (2.10) (b),

$$\Big(\frac{\partial \tilde F}{\partial y}\Big)^{-1} \trianglelefteq^{x_0}_y \Big(\Big(\frac{d\hat{\boldsymbol F}}{dy}\Big)_{\rm neg}\Big)^{-1} = \Big(\frac{d\hat{\boldsymbol F}_{\rm neg}}{dy}\Big)^{-1},$$

i.e., $\frac{\partial y}{\partial \tilde F} \trianglelefteq^{x_0}_y \frac{dy}{d\hat{\boldsymbol F}_{\rm neg}}$ and (2.21) holds for $i=1$. Inductively, by Lemma 2.3,

$$\begin{aligned} \frac{\partial^i y}{\partial \tilde{F}^i} &= \frac{\partial}{\partial \tilde F}\Big(\frac{\partial^{i-1} y}{\partial \tilde{F}^{i-1}}\Big) = \frac{\partial}{\partial y}\Big(\frac{\partial^{i-1} y}{\partial \tilde{F}^{i-1}}\Big)\Big(\frac{\partial \tilde F}{\partial y}\Big)^{-1} \\ &\trianglelefteq^{x_0}_y \frac{d}{dy}\Big(\frac{d^{i-1} y}{d\hat{\boldsymbol F}^{i-1}_{\rm neg}}\Big)\Big(\frac{d\hat{\boldsymbol F}_{\rm neg}}{dy}\Big)^{-1} = \frac{d^i y}{d\hat{\boldsymbol F}^i_{\rm neg}}. \end{aligned} \tag{2.22}$$

This proves (2.21). Using (2.21) and noting from (2.14) and (2.17), we have

$$\begin{aligned} \boldsymbol{b}_i &= \frac{1}{i!}\frac{\partial^i y}{\partial \tilde{F}^i}\Big|_{\tilde F = 0} = \frac{1}{i!}\frac{\partial^i y}{\partial \tilde{F}^i}\Big|_{y=0} \\ &\trianglelefteq^{x_0} \frac{1}{i!}\frac{d^i y}{d\hat{\boldsymbol F}^i_{\rm neg}}\Big|_{y=0} = \frac{1}{i!}\frac{d^i y}{d\hat{\boldsymbol F}^i_{\rm neg}}\Big|_{\hat{\boldsymbol F}_{\rm neg}=0} = \hat{\boldsymbol b}_i. \end{aligned}$$

This proves (2.19). Since $\tilde{F} \trianglelefteq_y^{x_0} \hat{\boldsymbol{F}}$ and $y^{\,\rm neg}$ is a controlling function, we have $y^{\,\rm neg}(\tilde{F}) \trianglelefteq_y^{x_0} y^{\,\rm neg}(\hat{\boldsymbol{F}})$. This together with (2.19) proves (2.20). $\square$

3. Proof of Theorem 1.3

First we use (1.13) to prove (1.14): Assume $h_{p_0,p_1} = |y_t|$, $n_t = |x_t|$ (for $t = 0$ or 1; the proof for the case $h_{p_0,p_1} = |x_t|$, $n_t = |y_t|$ is exactly similar). Then the only nontrivial case in (1.14) is the case when $a = |y_t|$, $b = |x_t|$. In this case, we have

$$|a-b| = \Big||y_t| - |x_t|\Big| \le |y_t + x_t|$$

$$< h_{p_0,p_1}^{\frac{m}{m+1}} = |y_t|^{\frac{m}{m+1}} < |x_t|^{\frac{m+1}{m+2}} = n_t^{\frac{m+1}{m+2}}, \tag{3.1}$$

where the last inequality follows from the fact that by (1.13), we have (when $|y_t| = h_{p_0,p_1} \ge \boldsymbol{s}_0$ is sufficiently large)

$$|x_t| > |y_t| - h_{p_0,p_1}^{\frac{m}{m+1}} = |y_t| - |y_t|^{\frac{m}{m+1}} > |y_t|^{\frac{m(m+2)}{(m+1)^2}}. \tag{3.2}$$

To prove (1.13), assume conversely that there exists $(p_{0i}, p_{1i}) = \big((x_{0i}, y_{0i}), (x_{1i}, y_{1i})\big) \in V$ for any $i \in \mathbb{Z}_{>0}$ satisfying

$$h_{p_{0i},p_{1i}} \ge i, \tag{3.3}$$

such that at least one of the following does not hold:

$$\text{(i)} \ \ |x_{0i} + y_{0i}| < h_{p_{0i},p_{1i}}^{\frac{m}{m+1}}, \qquad \text{(ii)} \ \ |x_{1i} + y_{1i}| < h_{p_{0i},p_{1i}}^{\frac{m}{m+1}}. \tag{3.4}$$

Thus we obtain a sequence (p_{0i}, p_{1i}), $i = 1, 2, ...$ Since at least one of the conditions in (3.4) cannot hold for infinite many i's, if necessary by replacing the sequence by a subsequence [if the sequence (p_{0i}, p_{1i}) is replaced by the subsequence (p_{0,i_j}, p_{1,i_j}), then we always have $i_j \ge j$; thus (3.3) still holds after the replacement], we may assume one of the conditions in (3.4) does not hold for all i. If necessary by switching p_{0i} and p_{1i}, we can assume (3.4) (i) cannot hold for all i, i.e.,

$$|x_{0i} + y_{0i}| \ge h_{p_{0i},p_{1i}}^{\frac{m}{m+1}} \to \infty, \tag{3.5}$$

for all $i \gg 1$. We need to use the following notations:

$$a_i \sim b_i, \quad a_i \prec b_i, \quad a_i \preceq b_i, \tag{3.6}$$

which mean respectively

$$\boldsymbol{s}_1 < \left|\frac{a_i}{b_i}\right| < \boldsymbol{s}_2, \qquad \lim_{i\to\infty} \frac{a_i}{b_i} = 0, \qquad \left|\frac{a_i}{b_i}\right| \le \boldsymbol{s}_1,$$

for some fixed $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbb{R}_{>0}$. By (1.5), we can write, for some $f_{jk} \in \mathbb{C}$,

$$F = F_L + F_1 \quad \text{with} \quad F_1 = \sum_{j=0}^{m-1} y^{m-1-j} \sum_{k=0}^{j+1} f_{jk} x^k. \tag{3.7}$$

Since $|x_{0i}|, |y_{0i}| \le h_{p_{0i},p_{1i}}$, by (3.5) and (3.7) (which shows that $\deg F_1 \le m-1$), we have

$$F_1(p_{0i}) \preceq h_{p_{0i},p_{1i}}^{m-1} \prec (x_{0i} + y_{0i})^m = F_L(p_{0i}), \tag{3.8}$$

and thus [we remark that although it is possible that $(x_{0i}+y_{0i})^m \prec h^m_{p_{0i},p_{1i}}$, it is very crucial that we have (3.9)]

$$F(p_{0i}) \sim F_L(p_{0i}) = (x_{0i}+y_{0i})^m. \tag{3.9}$$

Similarly, $F_1(p_{1i}) \preceq h^{m-1}_{p_{0i},p_{1i}} \prec (x_{0i}+y_{0i})^m = F_L(p_{0i})$. We obtain the following important fact:

$$\begin{aligned} 1 &= \frac{F(p_{1i})}{F(p_{0i})} = \lim_{i\to\infty} \frac{F(p_{1i})}{F(p_{0i})} \\ &= \lim_{i\to\infty} \frac{\frac{F_L(p_{1i})}{F_L(p_{0i})} + \frac{F_1(p_{1i})}{F_L(p_{0i})}}{1+\frac{F_1(p_{0i})}{F_L(p_{0i})}} = \lim_{i\to\infty} \frac{F_L(p_{1i})}{F_L(p_{0i})} = \lim_{i\to\infty} \frac{(x_{1i}+y_{1i})^m}{(x_{0i}+y_{0i})^m}. \end{aligned} \tag{3.10}$$

Therefore, by replacing the sequence by a subsequence, we have

$$\lim_{i\to\infty} \frac{x_{1i}+y_{1i}}{x_{0i}+y_{0i}} = \omega, \tag{3.11}$$

where ω is some m-th root of unity. Furthermore, when $i \gg 1$, by (3.5) we have [cf. Convention 2.1 (3)]

$$\mathcal{E} := \frac{h^{m-1}_{p_{0i},p_{1i}}}{\beta_0^m} \to 0, \qquad \text{where} \qquad \beta_0 := x_{0i}+y_{0i}. \tag{3.12}$$

Here and below, we remark that the notation "$a \to b$" means that a is sufficiently close to b. Set

$$\beta_1 := \frac{x_{1i}+y_{1i}}{x_{0i}+y_{0i}} - 1 \quad \to \quad \omega - 1. \tag{3.13}$$

Remark 3.1. Before continuing, we would like to remark that our idea is to take some variable change [cf. (3.16)] to send the leading part F_L of F to a "leading term" [cf. (3.20)], which has the highest absolute value when (x,y) is set to p_{0i} or p_{1i} [cf. (3.22)] so that when we expand it as a power series of y, it converges absolutely [cf. (3.25)], and further, the inverse function converges absolutely (cf. Lemma 3.3). Then we can derive a contradiction [cf. (3.49)].

Proof of Theorem 1.3. Now we begin our proof of (1.13) in Theorem 1.3 as follows. Since $x_{0i} \ne x_{1i}$ or $y_{0i} \ne y_{1i}$ for all i, replacing the sequence by a subsequence we may assume that either $x_{0i} \ne x_{1i}$ for all i or else $y_{0i} \ne y_{1i}$ for all i. By symmetry, we can assume $x_{0i} \ne x_{1i}$ for all i. Set [where $\boldsymbol{i} = \sqrt{-1}$, cf. the statement after (3.21) to see why we need to choose such a β_2]

$$u_1 = 1 + \beta_1 x + \beta_2 x(1-x) \in \mathbb{C}[x], \qquad \text{where} \quad \beta_2 = \begin{cases} 0 & \text{if } \omega \ne -1, \\ \boldsymbol{i} & \text{else.} \end{cases} \tag{3.14}$$

We have

Lemma 3.2. *There exists some $\delta > 0$ independent of $\mathcal{E}$ such that for all $a \in \mathbb{R}_{\ge 0}$ with $0 \le a \le 1$, when $i \gg 1$, we have*

$$|u_1(a)| > \delta. \tag{3.15}$$

Proof. Fix $\delta_1 \in \mathbb{R}_{>0}$ to be sufficiently small.

- First assume $\omega = 1$ (then $\beta_2 = 0$). By (3.13), we can then assume $|\beta_1| < \delta_1$. Then for a with $0 \le a \le 1$, we have $|u_1(a)| \ge 1 - |\beta_1| a \ge 1 - \delta_1$.

- Next assume $\omega = -1$ (then $\beta_2 = \boldsymbol{i}$). We can then assume $|\beta_{1\,\mathrm{im}}| < \delta_1^2$ [cf. Convention 2.1 (1)] and $2 - \delta_1^2 \le |\beta_{1\,\mathrm{re}}| \le 2 + \delta_1^2$ by (3.13). For a with $\delta_1 \le a \le 1 - \delta_1$, we have

$$|u_1(a)| \ge |u_1(a)_{\mathrm{im}}| = \big|\beta_{1\,\mathrm{im}}a + \beta_{2\,\mathrm{im}}a(1-a)\big| \ge a(1-a) - |\beta_{1\,\mathrm{im}}|a \ge \delta_1(1-\delta_1) - \delta_1^2.$$

  If $0 \le a \le \delta_1$, we have $|u_1(a)| \ge |u_1(a)_{\mathrm{re}}| = |1 + \beta_{1\,\mathrm{re}}a| \ge 1 - (2+\delta_1^2)\delta_1$. If $1 - \delta_1 \le a \le 1$, then $|u_1(a)| \ge |u_1(a)_{\mathrm{re}}| = |1 + \beta_{1\,\mathrm{re}}a| \ge (2-\delta_1)(1-\delta_1) - 1$.
- Now assume $\omega \ne \pm 1$ (then $\beta_2 = 0$). We can then assume $|\beta_{1\,\mathrm{im}}| \ge \delta_1$ and $|\beta_1| \le 2 + \delta_1$. If $0 \le a \le \delta_1$, we have $|u_1(a)| \ge 1 - |\beta_1|\delta_1 \ge 1 - (2+\delta_1)\delta_1$. If $\delta_1 \le a \le 1$, then $|u_1(a)| \ge |u_1(a)_{\mathrm{im}}| = |\beta_{1\,\mathrm{im}}|\delta_1 \ge \delta_1^2$.

In any case we can choose a unified $\delta > 0$ such that (3.15) holds. □

Now we fix a sufficiently large i [thus $\mathcal{E} > 0$ defined in (3.12) is sufficiently small]. Set [cf. Remark 3.1, our purpose is to use the variable change (3.17) to send the leading part F_L of F to the element (3.20) which is a term (the "leading term") with the lowest degree of y in $\hat{F}$, cf. (3.21)]

$$u = x_{0i} + \beta_3 x, \qquad v = \beta_0 u_1 y^{-1} - u, \qquad \beta_3 = x_{1i} - x_{0i}, \tag{3.16}$$

$$\hat{F} = \beta_0^{-m} F(u,v), \qquad \hat{G} = -\beta_0^{m-1}\beta_3^{-1} G(u,v) \in \mathbb{C}[x, u_1^{-1}, y^{\pm 1}] \subset \mathbb{C}(x)[y^{\pm 1}]. \tag{3.17}$$

Then one can easily verify that $J(u,v) = \frac{du}{dx}\frac{\partial v}{\partial y} = -\beta_0\beta_3 u_1 y^{-2}$ and

$$u|_{(x,y)=(0,1)} = x_{0i}, \quad u|_{(x,y)=(1,1)} = x_{1i}, \quad v|_{(x,y)=(0,1)} = y_{0i}, \quad v|_{(x,y)=(1,1)} = y_{1i}. \tag{3.18}$$

Thus we have, for $q_0 = (0,1)$, $q_1 = (1,1)$,

$$J(\hat{F},\hat{G}) = u_1 y^{-2}, \qquad \big(\hat{F}(q_0), \hat{G}(q_0)\big) = \big(\hat{F}(q_1), \hat{G}(q_1)\big). \tag{3.19}$$

Note that the leading part F_L of F contributes to $\hat{F}$ the following element (which is the only term in $\hat{F}$ with the lowest y-degree $-m$, referred to as the *leading term* of $\hat{F}$):

$$\beta_0^{-m} F_L|_{(x,y)=(u,v)} = \beta_0^{-m}(u+v)^m = u_1^m y^{-m}. \tag{3.20}$$

Since all coefficients of x and y^{-1} in u or v have absolute values being $\preceq$ the height $h_{p_{0i},p_{1i}}$ [cf. (3.16)], due to the factor β_0^{-m} in $\hat{F}$ [cf. (3.17)], we see from (3.7) and (3.12) that other terms of F (i.e., terms in F_1) can only contribute $O(\mathcal{E})^1$ elements to $\hat{F}$ [cf. Convention 2.1 (3) for notation $O(\mathcal{E})^j$]. Thus we can write, for some $f_j = u_1^{-m}\hat{f}_j \in \mathbb{C}[x, u_1^{-1}] \subset \mathbb{C}(x)$ with $\hat{f}_j = O(\mathcal{E})^1$,

$$\hat{F} = u_1^m y^{-m}\Big(1 + \sum_{j=1}^{m} f_j y^j\Big). \tag{3.21}$$

By (3.15), we see that $f_j(a)$ for $0 \le a \le 1$ is well-defined for any j and $f_j(a) = O(\mathcal{E})^1$ [this is why we need to choose some β_2 to satisfy (3.15)]. Set

$$\boldsymbol{s}_1 = \max\big\{|f_j(a)| \;\big|\; 1 \le j \le m,\, 0 \le a \le 1\big\} = O(\mathcal{E})^1. \tag{3.22}$$

Let $0 \le a \le 1$. Take [here we choose an m-th root of u_1^m to be u_1, this choice will not cause any problem since we will only encounter integral powers of u_1 below, cf. (2.3)]

$$P := \hat{F}^{-\frac{1}{m}} \in u_1^{-1}y + y^2\mathbb{C}(x)[[y]], \tag{3.23}$$

$$\hat{\boldsymbol{F}} = |u_1(a)|^m y^{-m}(1 + \hat{F}_-), \quad \text{where} \quad \hat{F}_- = \boldsymbol{s}_1 \sum_{j=1}^{m} y^j = O(\mathcal{E})^1. \tag{3.24}$$

We have (cf. Definition 2.2 and Lemma 2.3),

$$\hat{F} \trianglelefteq_y^a \hat{\boldsymbol{F}}, \qquad P \trianglelefteq_y^a \hat{\boldsymbol{P}} := |u_1(a)|^{-1}y(1 - \hat{F}_-)^{-\frac{1}{m}}. \tag{3.25}$$

Thus $\hat{F}, P$ converges absolutely [by Lemma 2.3 (3)] when setting $x = a$ and $y = 1$. Let

$$P_0 := P|_{(x,y)=(0,1)} = 1 + O(\mathcal{E})^1, \tag{3.26}$$

where the last equality can be easily seen from (3.21) and (3.23) by noting that $u_1(0) = 1$. Write [cf. (2.5) and (2.14)]

$$\text{(i) } y = u_1 P + \sum_{j=2}^{\infty} \boldsymbol{b}_j P^j, \qquad \text{(ii) } \hat{G} = \sum_{j=-m^G}^{\infty} c_j P^j, \tag{3.27}$$

for some $\boldsymbol{b}_i, c_i \in \mathbb{C}[x, u_1^{-1}]$, where we assume that $\hat{G}$ has the lowest y-degree $-m^G$. To continue the proof of Theorem 1.3, we need the following lemma. First, let $0 \le a \le 1$.

Lemma 3.3. (1) *The series in* (3.27) (i) *converges absolutely when setting* (x, P) *to* (a, P_0), *and*

$$Y_0(a) := y|_{(x,P)=(a,P_0)} = u_1(a) + O(\mathcal{E})^1. \tag{3.28}$$

(2) *Regarding* $\left(\frac{\partial \hat{F}}{\partial y}\right)^{-1}$ *as a series of* P, *it converges absolutely when setting* (x, P) *to* (a, P_0). *Furthermore,*

$$\left(\frac{\partial \hat{F}}{\partial y}\right)^{-1}\Big|_{(x,P)=(a,P_0)} = -m^{-1}u_1(a) + O(\mathcal{E})^1. \tag{3.29}$$

(3) *The series in* (3.27) (ii) *converges absolutely when setting* (x, P) *to* (a, P_0).

Proof. (1) (cf. Remark 3.4 for a simpler proof) Note that the negative correspondence of $\hat{\boldsymbol{P}}$ is [cf. (2.8)]

$$\hat{\boldsymbol{P}}_{\text{neg}} := 2|u_1(a)|^{-1}y - \hat{\boldsymbol{P}} = 2|u_1(a)|^{-1}y - |u_1(a)|^{-1}y(1 - F_-)^{-\frac{1}{m}}. \tag{3.30}$$

Let $y^{\text{neg}} = y^{\text{neg}}(\hat{\boldsymbol{P}}_{\text{neg}})$ be the inverse function of $\hat{\boldsymbol{P}}_{\text{neg}}$ [cf. (2.17)]. Then Lemma 2.4 shows that

$$y(P) \trianglelefteq_P^a y^{\text{neg}}(P). \tag{3.31}$$

Thus to see whether the series in (3.27) (i) [which is the left-hand side of (3.31)] converges absolutely when setting (x, P) to (a, P_0), it suffices to see if the series $y^{\text{neg}}(P)$ [which is the right-hand side of (3.31)] converges when setting P to $|P_0|$. The latter is equivalent to whether (3.30) has the solution for y when $\hat{\boldsymbol{P}}_{\text{neg}}$ is set to $|P_0|$ (note that the solution, if exists, must be unique by noting that the inverse function of $\hat{\boldsymbol{P}}_{\text{neg}}$ is a controlling function and a controlling function which is a nontrivial power series of y must be a strictly increasing function). Note from (3.26) that there exist some $w \in \mathbb{R}$ and some fixed numbers $\boldsymbol{s}_2, \boldsymbol{s}_3 \in \mathbb{R}_{>0}$ (i.e., $\boldsymbol{s}_2, \boldsymbol{s}_3$ are independent of $\mathcal{E}$) such that

$$|P_0| = 1 + w \quad \text{ with } \quad -\boldsymbol{s}_2\mathcal{E} \le w \le \boldsymbol{s}_3\mathcal{E}. \tag{3.32}$$

Consider the right-hand side of (3.30):

- if we set y to $|u_1(a)| - \boldsymbol{s}_4\mathcal{E}$ for some sufficiently large $\boldsymbol{s}_4$, then it obviously has some value $1 + w_1$ with $w_1 < -\boldsymbol{s}_2\mathcal{E} \le w$;
- if we set y to $|u_1(a)| + \boldsymbol{s}_5\mathcal{E}$ for some sufficiently large $\boldsymbol{s}_5$, then it has some value $1 + w_2$ with $w_2 > \boldsymbol{s}_3\mathcal{E} \ge w$.

Since the right-hand side of (3.30) is a continuous function on y, this shows that there exists (unique) $y_0 \in \mathbb{R}_{>0}$ such that

$$\hat{\boldsymbol{P}}_{\text{neg}}|_{y=y_0} = |P_0|, \qquad \text{and obviously,} \quad y_0 = |u_1(a)| + O(\mathcal{E})^1, \tag{3.33}$$

i.e., (3.30) has the solution $y = y_0$ when $\hat{\boldsymbol{P}}_{\text{neg}}$ is set to $|P_0|$, and thus the first part of (1) follows. As for (3.28), note that $Y_0(a)$ is the solution of y in the equation $P_0 = P|_{x=a}$. Using (3.28) in this equation, we see that it holds up to $O(\mathcal{E})^1$.

(2) By Lemmas 2.3 and 2.4, using (3.23)–(3.25), we have

$$\Big(\frac{\partial \hat{F}}{\partial y}\Big)^{-1} \trianglelefteq_y^a \frac{y^{m+1}}{m|u_1(a)|^m(1-Q_-)} \trianglelefteq_P \frac{y^{m+1}}{m|u_1(a)|^m(1-Q_-)}\Big|_{y=y^{\,\text{neg}}(P)}, \tag{3.34}$$

where Q_- is the following [then the part "$\trianglelefteq_y^a$" in (3.34) follows from (3.24); the second equality below follows from the fact in (3.22) that $\boldsymbol{s}_1 = O(\mathcal{E})^1$]

$$Q_- = \boldsymbol{s}_1 \sum_{j=1}^{m} \Big|\frac{m-j}{m}\Big| y^j = O(\mathcal{E})^1. \tag{3.35}$$

The right-hand side of (3.34) (a controlling function) converges obviously when setting P to $|P_0|$ since by (3.33), we have $y^{\,\text{neg}}(|P_0|) = y_0 = |u_1(a)| + O(\mathcal{E})^1$ and so by (3.35),

$$0 \le Q_-\big|_{y=y^{\,\text{neg}}(|P_0|)} = O(\mathcal{E})^1 < 1. \tag{3.36}$$

This proves the first statement of (2) [cf. (2.12)]. As for the second statement, note that setting (x, P) to (a, P_0) is equivalent to setting (x, y) to $\big(a, Y_0(a)\big)$. Then (3.29) follows from (3.21), (3.22) and (3.28).

(3) follows from (1) since $\hat{G}|_{x=a}$ is a polynomial on $y^{\pm 1}$ [cf. (3.44) and the statement after it]. This proves Lemma 3.3. □

Remark 3.4. Let $a \in \mathbb{R}$ with $0 \le a \le 1$. By (3.21), we can choose sufficiently small fixed $\eta \in \mathbb{R}_{>0}$ such that for $\mathcal{E}_1 = \mathcal{E}^\eta$, we have

$$\hat{F} \trianglelefteq_y^a |u_1(a)|^m y^{-m}\Big(1 + \sum_{j=1}^{\overline{m}} \mathcal{E}_1^j y^j\Big) \trianglelefteq_y |u_1(a)|^m y^{-m} \sum_{j=0}^{\infty} \mathcal{E}_1^j y^j = \frac{|u_1(a)|^m y^{-m}}{1 - \mathcal{E}_1 y}. \tag{3.37}$$

Thus by (2.10) (b), we have

$$P = \hat{F}^{-\frac{1}{m}} \trianglelefteq_y |u(a)|^{-1} y\Big(1 - \sum_{j=1}^{\infty} \mathcal{E}_1^j y^j\Big)^{-1} = |u(a)|^{-1} y\Big(1 - \frac{\mathcal{E}_1 y}{1 - \mathcal{E}_1 y}\Big)^{-1} = |u(a)|^{-1} \frac{(1 - \mathcal{E}_1 y) y}{1 - 2\mathcal{E}_1 y}. \tag{3.38}$$

Therefore we can in fact easily choose a simpler controlling function $\hat{\boldsymbol{P}}$ for P [cf. (3.25)]:

$$\hat{\boldsymbol{P}}=|u(a)|^{-1}\frac{(1-\mathcal{E}_1y)y}{1-2\mathcal{E}_1y}. \tag{3.39}$$

Then the negative correspondence of $\hat{\boldsymbol{P}}$ is simply the following,

$$\hat{\boldsymbol{P}}_{\rm neg}=2|u(a)|^{-1}y-\hat{\boldsymbol{P}}=|u(a)|^{-1}\frac{(1-3\mathcal{E}_1y)y}{1-2\mathcal{E}_1y}, \tag{3.40}$$

and we can explicitly write down the inverse function of $\hat{\boldsymbol{P}}_{\rm neg}$ by solving y from (3.40) to obtain $y^{\rm neg}(\hat{\boldsymbol{P}}_{\rm neg})$ [which, by Lemma 2.4 (1), must be a controlling function on $\hat{\boldsymbol{P}}_{\rm neg}$ (although it is not obvious to see)]

$$\text{(i) } y^{\rm neg}(\hat{\boldsymbol{P}}_{\rm neg})=\frac{1+\mathcal{E}_1t-A(t)}{6\mathcal{E}_1}\Big|_{t=\hat{\boldsymbol{P}}_{\rm neg}} \quad \text{with}$$

$$\text{(ii) } A(t)=\Big(1-8\mathcal{E}_1|u(a)|t+4\mathcal{E}_1^2|u(a)|^2t^2\Big)^{\frac12}=\Big(1-\alpha_1|u(a)|\mathcal{E}_1t\Big)^{\frac12}\Big(1-\alpha_2|u(a)|\mathcal{E}_1t\Big)^{\frac12}, \tag{3.41}$$

where $0<\alpha_1:=\frac{2}{2+\sqrt3}<\alpha_2:=\frac{2}{2-\sqrt3}$. By Lemma 2.3, we have

$$A(t)\trianglelefteq_t\Big(1-\alpha_1|u(a)|\mathcal{E}_1t\Big)^{-\frac12}\Big(1-\alpha_2|u(a)|\mathcal{E}_1t\Big)^{-\frac12}\trianglelefteq_t\Big(1-\alpha_2|u(a)|\mathcal{E}_1t\Big)^{-1}. \tag{3.42}$$

Using this, by comparing the coefficients of t^i (with $t=\hat{\boldsymbol{P}}_{\rm neg}$) in (3.41) (i) with that in the right-hand side of the following, we obtain

$$y^{\rm neg}(\hat{\boldsymbol{P}}_{\rm neg})\trianglelefteq_{\hat{\boldsymbol{P}}_{\rm neg}}|u_1(a)|\hat{\boldsymbol{P}}_{\rm neg}+\frac{\alpha_2^2|u_1(a)|^2\mathcal{E}_1\hat{\boldsymbol{P}}_{\rm neg}^2}{1-\alpha_2|u_1(a)|\mathcal{E}_1\hat{\boldsymbol{P}}_{\rm neg}}. \tag{3.43}$$

From this, one easily sees that the right-hand side of (3.31) converges when P is set to $|P_0|$ because the right-hand side of (3.43) converges when $\hat{\boldsymbol{P}}_{\rm neg}$ is set to $|P_0|$. Thus the proof of Lemma 3.3 (1) is easier (we have used the above proof in Lemma 3.3 as it can be adapted in some more general situation). Furthermore, by (3.31), (3.43) and Lemma 2.3, we have

$$\frac{1}{y(P)}\trianglelefteq_P\frac{1}{|u_1(a)|P\Big(1-\frac{\alpha_2^2|u_1(a)|\mathcal{E}_1P}{1-\alpha_2|u_1(a)|\mathcal{E}_1P}\Big)}. \tag{3.44}$$

From this we also see that when we regard y^{-1} as a series of P, it converges absolutely when setting (x,P) to (a,P_0).

Now we return to our proof of Theorem 1.3. By Lemma 3.3 (3), we are now safe to set (x,y) to $(0,1)$ and $(1,1)$ [which is equivalent to setting (x,P) to $(0,P_0)$ and $(1,P_0)$ respectively] in (3.27) (ii) to obtain

$$0=\hat{G}(1,1)-\hat{G}(0,1)=\sum_{j=-m^G}^{\infty}\big(c_j(1)-c_j(0)\big)P_0^j. \tag{3.45}$$

We will need the following very simple fact for any possible j,

$$c_j(1)-c_j(0)=\int_0^1\frac{dc_j(x)}{dx}dx, \tag{3.46}$$

which is obvious if $c_j(x)$ is a real function. To see this in general, noting that $c_j(x) \in \mathbb{C}[x, u_1^{-1}]$, we can write $c_j(x) = \hat{c}_{1j}(x) + \boldsymbol{i}\hat{c}_{2j}(x)$ with $\hat{c}_{tj}(x) = \frac{\tilde{c}_{tj}(x)}{(u_1\bar{u}_1)^{n_j}}$ for some $\tilde{c}_{tj}(x) \in \mathbb{R}[x]$ $(t = 1, 2)$, $n_j \in \mathbb{Z}_{>0}$ and $\bar{u}_1 = 1+\bar{\beta}_1 x+\bar{\beta}_2 x(1-x)$, where $\bar{\beta}_1, \bar{\beta}_2$ are the complex conjugate numbers of β_1, β_2 [cf. (3.14)]. Note that $u_1\bar{u}_1 \in \mathbb{R}[x]$ and $u_1\bar{u}_1|_{x=a} = |u_1(a)|^2 > \delta^2$ when $0 \le a \le 1$ by (3.15). Thus $\hat{c}_{tj}(x)$ is a well defined real rational function when $0 \le x \le 1$, and we have $\hat{c}_{tj}(1) - \hat{c}_{tj}(0) = \int_0^1 \frac{d\hat{c}_{tj}(x)}{dx}dx$. Thus (3.46) holds.

Denote

$$Q := -J(\hat{F}, \hat{G})\Big(\frac{\partial \hat{F}}{\partial y}\Big)^{-1} = -u_1 y^{-2}\Big(\frac{\partial \hat{F}}{\partial y}\Big)^{-1}, \tag{3.47}$$

where the last equality follows from (3.19). Take the Jacobian of $\hat{F}$ with (3.27) (ii), by (3.19) and (3.47), we obtain [by regarding Q as in $\mathbb{C}(x)((y))$],

$$Q = \sum_{j=-m^G}^{\infty} \frac{dc_j}{dx} P^j. \tag{3.48}$$

Since Q has the form (3.47), by Lemma 3.3, we see that when we expand Q as a power series on P [that is, (3.48)] and when P is set to P_0, the series [i.e, (3.48)] must converge absolutely and uniformly for $x \in [0, 1] := \{x \in \mathbb{R} \,|\, 0 \le x \le 1\}$. This together with (3.45), (3.46) and (3.48) implies

$$\begin{aligned} 0 &= \sum_{j=-m^G}^{\infty} (c_j(1) - c_j(0))P_0^j = \sum_{j=-m^G}^{\infty} \int_0^1 \frac{dc_j}{dx} P_0^j dx \\ &= \int_0^1 \sum_{j=-m^G}^{\infty} \frac{dc_j}{dx} P_0^j dx = \int_0^1 Q\,\big|_{P=P_0} dx = \int_0^1 Q\,\big|_{y=Y_0(x)} dx \\ &= m^{-1}\int_0^1 dx + O(\mathcal{E})^1 = m^{-1} + O(\mathcal{E})^1, \end{aligned} \tag{3.49}$$

which is a contradiction, where the sixth equality of (3.49) follows from (3.47), (3.28) and (3.29), and the fifth follows by noting that $Q\big|_{P=P_0}$ means that we need to express Q as an element in $\mathbb{C}(x)[[P]]$ [i.e., use (3.27) (i) to substitute y, that is exactly the equation (3.48)] then set P to P_0, which is equivalent to directly setting y to $Y_0(x)$ in Q [cf. (3.28)]. This means that if (3.3) holds, then we must have (3.4), i.e., we have Theorem 1.3. □

Remark 3.5. From the proof of Theorem 1.3, we see that it is enough to take, for example, $\boldsymbol{s}_0$ to be (where $m^G = \deg G$)

$$\boldsymbol{s}_0 = \alpha^{\alpha}, \qquad \text{where} \quad \alpha = 2(m + m^G)^2 + \sum_{i,j} \big(|\mathrm{C}_{\mathrm{oeff}}(F, x^i y^j)| + |\mathrm{C}_{\mathrm{oeff}}(G, x^i y^j)|\big). \tag{3.50}$$

4. Proof of Theorem 1.4

Proof of Theorem 1.4. Obviously, A_{k_0,k_1} defined in (1.16) is nonempty by assumption (1.15). To prove the boundness of A_{k_0,k_1}, assume, for $i = 1, 2, ...,$

$$(p_{0i}, p_{1i}) = \big((x_{0i}, y_{0i}), (x_{1i}, y_{1i})\big) \in A_{k_0,k_1}, \tag{4.1}$$

is a sequence such that the height $h_{p_{0i},p_{1i}} \to \infty$. By definition, we have $|x_{0i}| = k_0$, $|x_{1i}| = k_1$. Thus $|y_{0i}| = h_{p_{0i},p_{1i}}$ or $|y_{1i}| = h_{p_{0i},p_{1i}}$. In any case, at least one inequation of (1.13) is violated. Hence A_{k_0,k_1} is bounded.

To prove the closeness of A_{k_0,k_1}, let (4.1) be a sequence converging to some $(p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big) \in \mathbb{C}^4$. Then $\sigma(p_0) = \sigma(p_1)$ and $|x_0| = k_0$, $|x_1| = k_1$. We must have $p_0 \neq p_1$ [otherwise, the local bijectivity of σ does not hold at the point p_0, cf. arguments after (7.3)], i.e., $(p_0, p_1) \in A_{k_0,k_1}$, and so A_{k_0,k_1} is a closed set in $\mathbb{C}^4$, namely, we have Theorem 1.4 (i). From this, we see that γ_{k_0,k_1} in (1.17) is well-defined.

Now we prove Theorem 1.4 (iii). We will prove (1.18) (b) [the proof for (1.18) (a) is similar, but simpler, cf. Remark 4.2]. First we claim

$$\gamma_{0,0} > 0. \tag{4.2}$$

To see this, by definition, there exists $\big((0, \tilde y_0), (0, \tilde y_1)\big) \in A_{0,0}$ for some $\tilde y_0, \tilde y_1 \in \mathbb{C}$ with $\tilde y_0 \neq \tilde y_1$, thus also $\big((0, \tilde y_1), (0, \tilde y_0)\big) \in A_{0,0}$. By definition, $\gamma_{0,0} \geq \max\{|\tilde y_0|, |\tilde y_1|\} > 0$, i.e., we have (4.2).

Fix $k_0 > 0$. For any given $k_1' > 0$, let

$$\begin{aligned} \beta &= \max\{\gamma_{k_0,k_1} \,|\, k_1 \leq k_1'\} \\ &= \max\big\{\, |y_1| \;\big|\; (p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big) \in V,\ |x_0| = k_0,\ |x_1| \leq k_1' \,\big\}. \end{aligned} \tag{4.3}$$

Assume conversely that there exists $k_1 < k_1'$ with $\gamma_{k_0,k_1} = \beta$. We want to use the local bijectivity of Keller maps to obtain a contradiction. Let $\mathcal{E} > 0$ be a parameter such that $\mathcal{E} \to 0$ [cf. Convention 2.1 (3)]. Let

$$(\tilde p_0, \tilde p_1) = \big((\tilde x_0, \tilde y_0), (\tilde x_1, \tilde y_1)\big) \in V \quad \text{with} \quad |\tilde x_0| = k_0, \quad |\tilde x_1| = k_1, \quad |\tilde y_1| = \beta. \tag{4.4}$$

Set (and define $\tilde G_0, \tilde G_1$ similarly)

$$\tilde F_0 = F(\tilde x_0 + x, \tilde y_0 + y), \qquad \tilde F_1 = F(\tilde x_1 + x, \tilde y_1 + y). \tag{4.5}$$

Denote

$$\tilde a_0 = \mathrm{C_{oeff}}(\tilde F_0, x^1y^0), \quad \tilde b_0 = \mathrm{C_{oeff}}(\tilde F_0, x^0y^1), \quad \tilde a = \mathrm{C_{oeff}}(\tilde F_1, x^1y^0), \quad \tilde b = \mathrm{C_{oeff}}(\tilde F_1, x^0y^1). \tag{4.6}$$

We use $\tilde c_0, \tilde d_0, \tilde c, \tilde d$ to denote the corresponding elements for $\tilde G_0, \tilde G_1$. Then $A_0 = \big(\begin{smallmatrix} \tilde a_0 & \tilde c_0 \\ \tilde b_0 & \tilde d_0 \end{smallmatrix}\big)$ and $A = \big(\begin{smallmatrix} \tilde a & \tilde c \\ \tilde b & \tilde d \end{smallmatrix}\big)$ are invertible 2×2 matrices such that $\det A_0 = \det A = J(F, G) = 1$. For the purpose of proving Theorem 1.4 (iii), we can replace (F, G) by $(F, G)A_0^{-1}$, then A_0 becomes $A_0 = I_2$ (the 2×2 identity matrix), and AA_0^{-1} becomes the new A. Then we can write [here "$\equiv$" means equal modulo terms with degrees ≥ 3 by defining $\deg x = \deg y = 1$]

$$\begin{aligned} &\tilde F_0 \equiv x + \tilde\beta_1 x^2 + \tilde\beta_2 xy + \tilde\beta_3 y^2, \qquad &&\tilde F_1 \equiv \tilde a x + \tilde b y + \tilde\alpha_1 x^2 + \tilde\alpha_2 xy + \tilde\alpha_3 y^2, \\ &\tilde G_0 \equiv y + \tilde a_1 x^2 + \tilde a_2 xy + \tilde a_3 y^2, \qquad &&\tilde G_1 \equiv \tilde c x + \tilde d y + \tilde a_4 x^2 + \tilde a_5 xy + \tilde a_6 y^2, \end{aligned} \tag{4.7}$$

for some $\tilde a_i, \tilde\alpha_i, \tilde\beta_i \in \mathbb{C}$, where, by subtracting $\tilde F_i$ (resp., $\tilde G_i$) by the constant $\alpha_F = F(\tilde x_0, \tilde y_0)$ [resp., $\alpha_G = G(\tilde x_0, \tilde y_0)$], we have assumed $\tilde F_i$, $\tilde G_i$ do not contain constant terms.

For any $s, t, u, v \in \mathbb{C}$, denote

$$q_0 := (\dot{x}_0, \dot{y}_0) = (\tilde{x}_0 + s\mathcal{E}, \tilde{y}_0 + t\mathcal{E}), \qquad q_1 := (\dot{x}_1, \dot{y}_1) = (\tilde{x}_1 + u\mathcal{E}, \tilde{y}_1 + v\mathcal{E}). \tag{4.8}$$

The local bijectivity of Keller maps says that for any $u, v \in \mathbb{C}$ (cf. Remark 4.1), there exist $s, t \in \mathbb{C}$ such that $(q_0, q_1) \in V$, where (s, t) is uniquely determined from (u, v) by the equation

$$\Big(\tilde{F}_0(s\mathcal{E}, t\mathcal{E}), \tilde{G}_0(s\mathcal{E}, t\mathcal{E})\Big) = \Big(\tilde{F}_1(u\mathcal{E}, v\mathcal{E}), \tilde{G}_1(u\mathcal{E}, v\mathcal{E})\Big). \tag{4.9}$$

Remark 4.1. When we consider the local bijectivity of Keller maps, we always assume $u, v \in \mathbb{C}$ are bounded by some fixed $\boldsymbol{s} \in \mathbb{R}_{>0}$ (which is independent of $\mathcal{E}$, and we can assume $\mathcal{E}$ as small as we wish, for instance $\mathcal{E} < \boldsymbol{s}^{-\boldsymbol{s}^{\boldsymbol{s}}}$).

In fact we can easily use (4.7) to solve s, t up to $O(\mathcal{E})^1$ as follows,

$$s = s_0 + O(\mathcal{E})^1, \qquad s_0 = \tilde{a}u + \tilde{b}v. \tag{4.10}$$

We want to choose suitable u, v such that

$$\text{(i)}\ \ |\dot{x}_0| = |\tilde{x}_0 + s\mathcal{E}| = |\tilde{x}_0| \quad (= k_0 > 0\,), \qquad \text{(ii)}\ \ |\dot{y}_1| = |\tilde{y}_1 + v\mathcal{E}| > |\tilde{y}_1| \quad (= \beta\,). \tag{4.11}$$

First assume $\tilde{a} \neq 0$ in (4.10). Then we can easily first choose v to satisfy (4.11) (ii), then choose u to satisfy (4.11) (i) [cf. (4.10); we can also regard s as a free variable and solve $u = \tilde{a}^{-1}(s - \tilde{b}v) + O(\mathcal{E})^1$ from (4.10), and then using the fact that $x_0 \neq 0$ we can solve s from (4.11) (i) as in (4.16) below]. Since $|x_1| = k_1 < k_1'$, we automatically have $|\dot{x}_1| < k_1'$ [as $\mathcal{E} \ll 1$, cf. (4.8)]. This means that we can choose $(q_0, q_1) \in V$ with $|\dot{x}_0| = k_0$, $|\dot{x}_1| < k_1'$, but $|\dot{y}_1| > \beta$, which is a contradiction with the definition of β in (4.3).

Now assume $\tilde{a} = 0$ (and so $\tilde{b} \neq 0$, $\tilde{c} \neq 0$). In this case the situation is more complicated.

Remark 4.2. Before continuing, we remark that the proof of (1.18) (a) is easier: in that case condition (4.11) (i) should be replaced by the condition $|\tilde{x}_1 + u\mathcal{E}| = |\tilde{x}_1|$, which can be easily satisfied even in case $k_1 = 0$ (i.e., $\tilde{x}_1 = 0$). Thus (1.18) (a) holds.

Now we continue our proof. Since $|\tilde{x}_0| = k_0 > 0$, and $|\tilde{y}_1| = \beta \geq \gamma_{k_0,0} > \gamma_{0,0} > 0$ [where the first inequality follows from the definition of β in (4.3), the second from (1.18) (a) (cf. Remark 4.2), and the last from (4.2)], we can rewrite (4.11) as [cf. (4.10)]

$$\text{(i)}'\ |1 + \hat{s}\mathcal{E}| = 1, \qquad \text{(ii)}'\ |1 + \hat{v}\mathcal{E}| > 1, \qquad \text{where} \quad \hat{v} = \tilde{y}_1^{-1}v, \quad \hat{s} = \tilde{x}_0^{-1}s, \tag{4.12}$$

and regard $\hat{v}$ as a new variable. In order for the convenience to make use of $\hat{v}$, $\hat{s}$, we set [see also arguments after (5.10)],

$$\hat{F}_0 = \tilde{x}_0^{-1}\tilde{F}_0(\tilde{x}_0 x, y), \quad \hat{G}_0 = \tilde{G}_0(\tilde{x}_0 x, y), \quad \hat{F}_1 = \tilde{x}_0^{-1}\tilde{F}_1(x, \tilde{y}_1 y), \quad \hat{G}_1 = \tilde{G}_1(x, \tilde{y}_1 y), \tag{4.13}$$

and rewrite [cf. (4.7); we now use b, c, d, which are different from $\tilde{b}, \tilde{c}, \tilde{d}$ in (4.7), to denote the coefficients of linear parts of $\hat{F}_1, \hat{G}_1$]

$$\hat{F}_0 = \sum_{i\geq 2} a_i y^i + x\Big(1 + \sum_{i\geq 1} \hat{a}_i y^i\Big) + \cdots, \qquad \hat{F}_1 = \sum_{i\geq 2} b_i z^i + by\Big(1 + \sum_{i\geq 1} \hat{b}_i z^i\Big) + \cdots,$$

$$\hat{G}_0 = y + \sum_{i\geq 2} c_i y^i + \cdots, \qquad \hat{G}_1 = z + \sum_{i\geq 2} d_i z^i + \cdots, \qquad \text{where} \ \ z = cx + dy, \tag{4.14}$$

for some $a_i, \hat{a}_i, b_i, \hat{b}_i, c_i, d_i \in \mathbb{C}$, and where we regard $\hat{F}_1, \hat{G}_1$ as polynomials on y, z and we omit terms with x-degree ≥ 2 in $\hat{F}_0$ (or ≥ 1 in $\hat{G}_0$), and omit terms with y-degree ≥ 2 in $\hat{F}_1$ (or ≥ 1 in $\hat{G}_1$), which will be irrelevant to our computations below. In this case, by (4.14), we can solve

$$\hat{s} = b\hat{v} + O(\mathcal{E})^1. \tag{4.15}$$

If $b_{\rm im} \neq 0$ [cf. Convention 2.1 (1)], we can always choose suitable $\hat{v} \in \mathbb{C}$ with $\hat{v}_{\rm re} > 0$ such that both (i)$'$ and (ii)$'$ in (4.12) hold. Alteratively, we can also regard $\hat{s}$ as a free variable [and solve $\hat{v} = b^{-1}\hat{s} + O(\mathcal{E})^1$ from (4.15)] and determine $\hat{s}$ by solving $\hat{s}_{\rm re}$ from (4.12) (i)$'$ to obtain

$$\hat{s}_{\rm re} = \frac{-1 + (1 - \hat{s}_{\rm im}^2\mathcal{E}^2)^{\frac{1}{2}}}{\mathcal{E}} = -\frac{\hat{s}_{\rm im}^2\mathcal{E}}{2} + O(\mathcal{E})^3, \tag{4.16}$$

then choose $\hat{s}_{\rm im}$ [with $(b^{-1}\hat{s})_{\rm re} = \frac{b_{\rm re}\hat{s}_{\rm re} + b_{\rm im}\hat{s}_{\rm im}}{|b|^2} = \frac{b_{\rm im}\hat{s}_{\rm im}}{|b|^2} + O(\mathcal{E})^1 > 0$] to satisfy (4.12) (ii)$'$.

Now assume $b \in \mathbb{R}_{\neq 0}$. We claim that for at least one $i \geq 2$, we have

$$(a_i, c_i) \neq (b_i, d_i). \tag{4.17}$$

Otherwise we would in particular obtain (and the like for G)

$$F(\tilde{x}_0, \tilde{y}_0 + \boldsymbol{k}) = \tilde{x}_0\hat{F}_0\big|_{(x,y)=(0,\boldsymbol{k})} = \tilde{x}_0\hat{F}_1\big|_{(x,y)=(\boldsymbol{k}\,c^{-1},0)} = F(\tilde{x}_1 + \boldsymbol{k}c^{-1}, \tilde{y}_1), \quad \text{i.e.,} \tag{4.18}$$

$$\sigma(\hat{p}_0) = \sigma(\hat{p}_1) \quad \text{with}$$

$$\hat{p}_0 = (\hat{x}_0, \hat{y}_0) = (\tilde{x}_0, \tilde{y}_0 + \boldsymbol{k}), \qquad \hat{p}_1 = (\hat{x}_1, \hat{y}_1) = (\tilde{x}_1 + \boldsymbol{k}c^{-1}, \tilde{y}_1), \tag{4.19}$$

for all $\boldsymbol{k} \gg 1$ [cf. Convention 2.1 (3)]. Then $h_{\hat{p}_0,\hat{p}_1} \sim \boldsymbol{k}$ [when $\boldsymbol{k} \gg 1$, cf. (3.6)], and $|\hat{x}_1 + \hat{y}_1| \sim \boldsymbol{k} \succ h_{\hat{p}_0,\hat{p}_1}^{\frac{m}{m+1}}$, a contradiction with (1.13). Thus (4.17) holds. Let $i_0 \geq 2$ be the minimal i satisfying (4.17). By replacing $\hat{F}_j$ by $\hat{F}_j + \sum_{i=2}^{2i_0} \beta_i \hat{G}_j^i$ for some $\beta_i \in \mathbb{C}$ and $j = 0, 1$, thanks to the term y in $\hat{G}_0$, we can then suppose, for $2 \leq i \leq 2i_0$,

$$a_i = 0. \tag{4.20}$$

Now we need to consider two cases.

Case 1: *Assume* $b_k \neq 0$ *for some* $k \leq 2i_0$. Take minimal such $k \geq 2$. Setting [noting from (4.14) that this amounts to setting $y = \hat{v}\mathcal{E} = \check{v}\mathcal{E}^k$, $z = w\mathcal{E}$ in $\hat{F}_1$, $\hat{G}_1$, and setting $x = \hat{s}\mathcal{E}$, $y = t\mathcal{E}$ in $\hat{F}_0$, $\hat{G}_0$, and letting $\hat{F}_0 = \hat{F}_1$, $\hat{G}_0 = \hat{G}_1$ to solve $\hat{s}$, t],

$$\hat{v} = \check{v}\mathcal{E}^{k-1}, \qquad u = c^{-1}w - c^{-1}d\,\check{v}\mathcal{E}^{k-1}, \tag{4.21}$$

and regarding $\check{v}, w$ as new variables, we can then solve from (4.14) [cf. (4.10), (4.12) and (4.15); observe that all omitted terms and all coefficients $\hat{a}_i$'s, $\hat{b}_i$'s do not contribute to our solution of $\hat{s}$ up to $\mathcal{E}^k$] to obtain,

$$\hat{s} = (b\check{v} + b_k w^k)\mathcal{E}^{k-1} + O(\mathcal{E})^k. \tag{4.22}$$

Using this and the first equation of (4.21) in (4.12), one can then easily see that (4.11) have solutions [by taking, for example, $\check{v} > 0$ so that (ii)$'$ holds and then choosing w to satisfy (i)$'$].

Case 2: *Assume* $b_i = 0$ *for* $0 \leq i \leq 2i_0$. By computing the following coefficients, for $i \geq 1$,

$$\mathrm{C}_{\rm oeff}\big(J(\hat{F}_0, \hat{G}_0), x^0y^i\big) = 0 = \mathrm{C}_{\rm oeff}\big(J(\hat{F}_1, \hat{G}_1), x^0y^i\big), \tag{4.23}$$

and induction on i for $1 \le i < i_0$, one can easily obtain [using (4.20) and noting that i_0 is the minimal i satisfying (4.17)],

$$\hat{a}_i = \hat{b}_i \ \text{ for } \ i < i_0, \quad \text{ and } \quad \hat{a}_{i_0} \ne \hat{b}_{i_0}. \tag{4.24}$$

In this case, by setting [the first equation below means that $\check{v}\mathcal{E}$ contributes a positive $O(\mathcal{E})^{2i_0}$ element to the left-hand side of (4.12) (ii)′ since it does not have a real part, in particular (4.12) (ii)′ holds],

$$\hat{v} = v_1 \boldsymbol{i} \mathcal{E}^{i_0 - 1}, \qquad u = c^{-1} w - c^{-1} d v_1 \boldsymbol{i} \mathcal{E}^{i_0 - 1}, \tag{4.25}$$

for $v_1 \in \mathbb{R}_{\ne 0}$, we can then solve from (4.14) to obtain, for some nonzero $b'' \in \mathbb{C}$ [by (4.24); all omitted terms in (4.14) do not contribute to our solution of $\hat{s}$ up to $\mathcal{E}^{2i_0}$],

$$\hat{s}\mathcal{E} = b v_1 \boldsymbol{i} \mathcal{E}^{i_0} + b'' v_1 \boldsymbol{i} w^{i_0} \mathcal{E}^{2i_0} + O(\mathcal{E})^{2i_0+1}. \tag{4.26}$$

Since $b \in \mathbb{R}_{\ne 0}$, we see that (4.26) can only contribute an $O(\mathcal{E})^{2i_0}$ element to (4.12) (i)′. Using (4.26) and the first equation of (4.25) in (4.12), one can again see that (4.11) have solutions by choosing suitable w. This proves Theorem 1.4. □

5. Proof of Theorem 1.5 (1)

To prove Theorem 1.5 (1), let us make the following assumption [cf. Remark 1.6 (3)].

Assumption 5.1. *Assume Theorem* 1.5 (1) *is not true.*

Under this assumption, we have

Lemma 5.2. *For any* $\delta \in \mathbb{R}_{\ge 0}$, $k, k_0, k_1 \in \mathbb{R}_{>0}$ *with* $k > 1$, $\delta < \frac{1}{m}$, *we have* $\gamma_{k^{1+\delta}k_0, kk_1} < k\gamma_{k_0,k_1}$.

Proof. Assume the result is not true, then by choosing δ' with $\delta < \delta' < \frac{1}{m}$ and by Theorem 1.4 (iii), we may assume $\gamma_{\bar{k}^{1+\delta'}k_0, \bar{k}k_1} > \bar{k}\gamma_{k_0,k_1}$ for some $\bar{k}, k_0, k_1 \in \mathbb{R}_{>0}$ with $\bar{k} > 1$. Thus we can choose sufficiently small $\delta_1, \delta_2 \in \mathbb{R}_{>0}$ with $\delta_1 < \delta'$ satisfying (the following holds when $\delta_1 = \delta_2 = 0$ thus also holds when $\delta_1, \delta_2 > 0$ are sufficiently small)

$$\frac{k_1^{1+\delta_1}\big(\gamma_{\bar{k}^{1+\delta'}k_0, \bar{k}k_1} + \delta_2\big)}{(\bar{k}k_1)^{1+\delta_1}} > \gamma_{k_0,k_1} + \delta_2. \tag{5.1}$$

Take $\boldsymbol{k} \gg 1$. We define V_0 to be the subset of V consisting of elements $(p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big)$ satisfying [our aim is to design the following to satisfy Theorem 1.5 (1)]

$$\begin{aligned} &\text{(a) } 1 \le \big(k_1^{-1}|x_1|\big)^{1+\delta'-\boldsymbol{k}^{-3}} \le k_0^{-1}|x_0| \le \big(k_1^{-1}|x_1|\big)^{1+\delta'+\boldsymbol{k}^{-3}} \le \boldsymbol{k}^{1+\delta'+\boldsymbol{k}^{-3}}, \\ &\text{(b) } \frac{|y_1| + \delta_2}{|x_1|^{1+\delta_1}} \ge \frac{\gamma_{\bar{k}^{1+\delta'}k_0, \bar{k}k_1} + \delta_2}{(\bar{k}k_1)^{1+\delta_1}}. \end{aligned} \tag{5.2}$$

Then we can rewrite the above as the form in (1.19), and obviously we have (1.21). Further, by definition, there exists

$$(\check{p}_0, \check{p}_1) = \big((\check{x}_0, \check{y}_0), (\check{x}_1, \check{y}_1)\big) \in V \quad \text{with } \ |\check{x}_0| = \bar{k}^{1+\delta'}k_0, \ \ |\check{x}_1| = \bar{k}k_1, \ \ |\check{y}_1| = \gamma_{\bar{k}^{1+\delta'}k_0, \bar{k}k_1}. \tag{5.3}$$

Then one can easily see that $(\check{p}_0, \check{p}_1) \in V_0$, i.e., $V_0 \ne \emptyset$.

Let $(p_0,p_1)\in V_0$. In (5.2) (a), if the equality occurs in the first inequality, or two equalities simultaneously occur in the second and third inequalities, then we obtain that $|x_1|=k_1$, $|x_0|=k_0$, but by (5.1), (5.2) (b), the definition of γ_{k_0,k_1} and Theorem 1.4 (iii), we have

$$\gamma_{k_0,k_1}=\gamma_{|x_0|,|x_1|}\geq|y_1|\geq\frac{|x_1|^{1+\delta_1}\big(\gamma_{\bar k^{1+\delta'}k_0,\bar k k_1}+\delta_2\big)}{(\bar k k_1)^{1+\delta_1}}-\delta_2$$
$$=\frac{k_1^{1+\delta_1}\big(\gamma_{\bar k^{1+\delta'}k_0,\bar k k_1}+\delta_2\big)}{(\bar k k_1)^{1+\delta_1}}-\delta_2>\gamma_{k_0,k_1}, \tag{5.4}$$

which is a contradiction.

If the equality occurs in the last inequality of (5.2) (a), then one obtains that $|x_1|\sim\boldsymbol{k}$, $|x_0|\sim\boldsymbol{k}^{1+\delta'}$ when $\boldsymbol{k}\gg1$ [cf. (3.6); see Remark 5.3; note that $\boldsymbol{k}^{\boldsymbol{k}^{-3}}=1+O(\boldsymbol{k}^{-1})^3$]. By (5.2) (b), we have $|y_1|\succeq|x_1|^{1+\delta_1}\sim\boldsymbol{k}^{1+\delta_1}$. Note that (1.14) in particular implies that either $h_{p_0,p_1}\sim|x_0|\sim|y_0|$ or $h_{p_0,p_1}\sim|x_1|\sim|y_1|$, in any case we obtain that $h_{p_0,p_1}\preceq\boldsymbol{k}^{1+\delta'}$ (in fact the latter case cannot occur as $|x_1|=\boldsymbol{k}\prec\boldsymbol{k}^{1+\delta_1}\preceq|y_1|$). We have (where the part "$\succ$" follows by noting from $\delta'<\frac{1}{m}$ that $|y_1|\succeq\boldsymbol{k}^{1+\delta_1}\succ\boldsymbol{k}\succeq\boldsymbol{k}^{\frac{(1+\delta')m}{m+1}}$)

$$|x_1+y_1|\geq|y_1|-|x_1|\sim|y_1|\succ\boldsymbol{k}^{\frac{(1+\delta')m}{m+1}}\succeq h_{p_0,p_1}^{\frac{m}{m+1}}, \tag{5.5}$$

a contradiction with (1.13). This shows that Theorem 1.5 (1) holds, a contradiction with Assumption 5.1. The lemma is proven. □

Remark 5.3. Note that when we design the system (5.2), $\boldsymbol{k}$ is simply some fixed positive real number. When we say $\boldsymbol{k}\gg1$, it means that we may need to choose sufficiently large $\boldsymbol{k}$ such that the system (5.2) can satisfy our requirement. This will also apply to some similar situations later.

Lemma 5.4. *For any $k_0,k_1\in\mathbb{R}_{>0}$, we have $\gamma_{k_0,k_1}>k_1$.*

Proof. Assume the result is not true, then by choosing $k_0'\in\mathbb{R}_{>0}$ with $k_0'<k_0$, we may assume

$$\gamma_{k_0',k_1}<k_1, \tag{5.6}$$

for some $k_0',k_1>0$. Denote $\alpha=\frac{\gamma_{k_0',k_1}}{k_1}<1$ by (5.6). By Lemma 5.2, we have $\gamma_{\boldsymbol{k}k_0',\boldsymbol{k}k_1}<\boldsymbol{k}\gamma_{k_0',k_1}=\boldsymbol{k}k_1\alpha$ for all $\boldsymbol{k}\gg1$. Let

$$(p_0,p_1)\in V\quad\text{with}\quad|x_0|=\boldsymbol{k}k_0',\quad|x_1|=\boldsymbol{k}k_1,\quad|y_1|=\gamma_{\boldsymbol{k}k_0',\boldsymbol{k}k_1}<\boldsymbol{k}k_1\alpha. \tag{5.7}$$

Then as in the proof of the previous lemma, we have $h_{p_0,p_1}\sim\boldsymbol{k}$ when $\boldsymbol{k}\gg1$, but then

$$|x_1+y_1|\geq|x_1|-|y_1|>(1-\alpha)\boldsymbol{k}k_1>h_{p_0,p_1}^{\frac{m}{m+1}}, \tag{5.8}$$

which is a contradiction with (1.13). This proves Lemma 5.4. □

Now we fix sufficiently large $\boldsymbol{k}\gg1$. Take

$$(\bar p_0,\bar p_1)=\big((\bar x_0,\bar y_0),(\bar x_1,\bar y_1)\big)\in A_{\boldsymbol{k},\boldsymbol{k}}\quad\text{with}\quad|\bar x_0|=|\bar x_1|=\boldsymbol{k},\quad|\bar y_1|=\gamma_{\boldsymbol{k},\boldsymbol{k}}>\boldsymbol{k}, \tag{5.9}$$

where the inequality follows from Lemma 5.4. Similar to (4.5) (but not exactly), we define

$$F_0=F\big(\bar x_0(1+x),\bar y_0+y\big),\qquad F_1=F\big(\bar x_1(1+x),\bar y_1(1+y)\big), \tag{5.10}$$

and define G_0, G_1 similarly [thus the matrices A_0, A defined after (4.6) now have determinants $\det A_0 = \bar{x}_0 J(F,G) \neq 0$, $\det A = \bar{x}_1 \bar{y}_1 J(F,G) \neq 0$, and again by replacing (F_i, G_i) by $(F_i, G_i)A_0^{-1}$ for $i = 0, 1$, we can assume $A_0 = I_2$]. Similar to (4.7), we can write [from now on, we only need the linear parts of F, G],

$$F_0 \equiv x, \qquad F_1 \equiv -a_{\boldsymbol{k}} x + b_{\boldsymbol{k}} y, \tag{5.11}$$

$$G_0 \equiv y, \qquad G_1 \equiv cx + dy, \tag{5.12}$$

where we have written the coefficients of x, y in F_1 as $-a_{\boldsymbol{k}}, b_{\boldsymbol{k}}$ to emphasis that they may depend on $\boldsymbol{k}$ (of course other coefficients also depend on $\boldsymbol{k}$) and that $a_{\boldsymbol{k}}, b_{\boldsymbol{k}}$ are in fact positive as shown in the next lemma.

We define q_0, q_1 accordingly [similar to, but a slightly different from, (4.8), simply due to the different definitions in (5.10) and (4.5); we emphasis that the choice of $\mathcal{E}$ depend on $\boldsymbol{k}$: in general the larger $\boldsymbol{k}$ is, the smaller $\mathcal{E}$; but in any case once $\boldsymbol{k}$ is chosen we can always choose sufficiently small $\mathcal{E}$, cf. also Remark 4.1],

$$q_0 := (\dot{x}_0, \dot{y}_0) = \big(\bar{x}_0(1 + s\mathcal{E}), \bar{y}_0 + t\mathcal{E}\big), \qquad q_1 := (\dot{x}_1, \dot{y}_1) = \big(\bar{x}_1(1 + u\mathcal{E}), \bar{y}_1(1 + v\mathcal{E})\big). \tag{5.13}$$

In particular, we have as in (4.10),

$$s = -a_{\boldsymbol{k}} u + b_{\boldsymbol{k}} v + O(\mathcal{E})^1. \tag{5.14}$$

The numbers $a_{\boldsymbol{k}}, b_{\boldsymbol{k}}$ will play very crucial roles in our proofs in this section.

Lemma 5.5. *We have $a_{\boldsymbol{k}} > 0$, $b_{\boldsymbol{k}} > 0$.*

Proof. First assume $a_{\boldsymbol{k}\,\mathrm{im}} \neq 0$ or $a_{\boldsymbol{k}\,\mathrm{re}} < 0$ or $b_{\boldsymbol{k}\,\mathrm{im}} \neq 0$ or $b_{\boldsymbol{k}\,\mathrm{re}} < 0$ [cf. Convention 2.1 (1)]. Then from (5.14) one can easily choose u, v [with $u_{\mathrm{im}} \neq 0$, $u_{\mathrm{re}} < 0$, $v_{\mathrm{im}} \neq 0$, $v_{\mathrm{re}} > 0$ such that either $(a_{\boldsymbol{k}} u)_{\mathrm{re}} > 0$ or $(b_{\boldsymbol{k}} v)_{\mathrm{re}} < 0$, and so $s_{\mathrm{re}} < 0$] satisfying [cf. (5.13) and (5.14)],

$$\begin{aligned} &0 < k_0 := |\dot{x}_0| = \boldsymbol{k}|1 + s\mathcal{E}| < \boldsymbol{k}, \qquad 0 < k_1 := |\dot{x}_1| = \boldsymbol{k}|1 + u\mathcal{E}| < \boldsymbol{k}, \\ &|\dot{y}_1| = \gamma_{\boldsymbol{k},\boldsymbol{k}}|1 + v\mathcal{E}| > \gamma_{\boldsymbol{k},\boldsymbol{k}}, \end{aligned} \tag{5.15}$$

i.e., $0 < k_0 < \boldsymbol{k}$ and $0 < k_1 < \boldsymbol{k}$ with $\gamma_{k_0,k_1} \geq |\dot{y}_1| > \gamma_{\boldsymbol{k},\boldsymbol{k}}$, a contradiction with Theorem 1.4 (iii). Thus $a_{\boldsymbol{k}} \geq 0$ and $b_{\boldsymbol{k}} \geq 0$.

If $a_{\boldsymbol{k}} = 0$, similar to arguments after (4.12) [see also arguments after (7.7)], we may have two possible cases [cf. (4.21), (4.22) and (4.25), (4.26)]:

$$v = \hat{v}\mathcal{E}^{k-1}, \qquad u = c^{-1}w - c^{-1}d\,\hat{v}\mathcal{E}^{k-1}, \qquad s = (b_{\boldsymbol{k}}\hat{v} + b' w^k)\mathcal{E}^{k-1} + O(\mathcal{E})^k, \tag{5.16}$$

$$v = v_1 \boldsymbol{i}\mathcal{E}^{i_0-1}, \quad u = c^{-1}w - c^{-1}dv_1\boldsymbol{i}\mathcal{E}^{i_0-1}, \quad s\mathcal{E} = b_{\boldsymbol{k}} v_1 \boldsymbol{i}\mathcal{E}^{i_0} + b'' v_1 \boldsymbol{i} w^{i_0}\mathcal{E}^{2i_0} + O(\mathcal{E})^{2i_0+1}, \tag{5.17}$$

where $\hat{v}, w \in \mathbb{C}$, $v_1 \in \mathbb{R}_{\neq 0}$, $b', b'' \in \mathbb{C}_{\neq 0}$, $k, i_0 \in \mathbb{Z}_{\geq 2}$. Assume we have the case (5.16) [the proof for the case (5.17) is similar], we can first choose $\hat{v}$ with $\hat{v}_{\mathrm{re}} > 0$ so that the last inequation of (5.15) holds, then choose w with $(c^{-1}w)_{\mathrm{re}} < -1$ (sufficiently smaller than -1) and $(b'w^k)_{\mathrm{re}} < 0$ (sufficiently smaller than -1, such w can be always chosen since $k \geq 2$) such that the first two inequations of (5.15) hold. Thus (5.15) holds, and as before we obtain a contradiction. Therefore $a_{\boldsymbol{k}} > 0$. Similarly $b_{\boldsymbol{k}} > 0$. The lemma is proven. □

Lemma 5.6. *For any fixed $N\in\mathbb{R}_{>0}$, let $\delta'\in\mathbb{R}_{>0}$ be such that $\delta'>\ln(\boldsymbol{k})^{-N}$ (where $\ln(\cdot)$ is the natural logarithmic function), we have $\boldsymbol{k}<\gamma_{\boldsymbol{k},\boldsymbol{k}}<(1+\delta')\boldsymbol{k}$.*

Proof. By (1.14), when $\boldsymbol{k}\gg 1$, we either have $h_{\bar p_0,\bar p_1}\sim|x_0|\sim|y_0|$ or $h_{\bar p_0,\bar p_1}\sim|x_1|\sim|y_1|$. In any case, $h_{\bar p_0,\bar p_1}\sim\boldsymbol{k}$. If $\gamma_{\boldsymbol{k},\boldsymbol{k}}\ge(1+\delta')\boldsymbol{k}$, then when $\boldsymbol{k}\gg1$,

$$|\bar x_1+\bar y_1|\ge|\bar y_1|-|\bar x_1|\ge\delta'\boldsymbol{k}\ge\ln(\boldsymbol{k})^{-N}\boldsymbol{k}\succ\boldsymbol{k}^{\frac{m}{m+1}}\sim h_{\bar p_0,\bar p_1}^{\frac{m}{m+1}},\tag{5.18}$$

a contradiction with (1.13). □

Lemma 5.7. *For any fixed $\delta\in\mathbb{R}_{>0}$ with $\delta<\frac{1}{m}$, we have $b_{\boldsymbol{k}}\ge1+\delta+a_{\boldsymbol{k}}$ for all $\boldsymbol{k}>0$.*

Proof. Assume the lemma does not hold, then we can choose sufficiently small $\delta_1>0$ (which can depend on $\boldsymbol{k}$) such that

$$(1+\delta_1)b_{\boldsymbol{k}}<1+\delta-\delta_1+a_{\boldsymbol{k}}.\tag{5.19}$$

Let $\ell\gg\boldsymbol{k}$ (we can assume $\mathcal{E}<\ell^{-\ell}$, cf. Remark 4.1). We define V_0 to be the subset of V consisting of elements $(p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)$ satisfying [again our purpose is to design the following to satisfy Theorem 1.5 (1)]

$$\text{(i) } 1\le\big(\boldsymbol{k}^{-1}|x_1|\big)^{1+\delta-\delta_1-\ell^{-3}}\le\boldsymbol{k}^{-1}|x_0|\le\big(\boldsymbol{k}^{-1}|x_1|\big)^{1+\delta}\le\ell^{1+\delta},\quad\text{(ii) }\frac{\gamma_{\boldsymbol{k},\boldsymbol{k}}^{-1}|y_1|+\mathcal{E}^3}{\big(\boldsymbol{k}^{-1}|x_1|\big)^{1+\delta_1}}\ge1+\mathcal{E}^2.\tag{5.20}$$

Then we have (1.19) and (1.21).

Remark 5.8. Recall from statements inside the bracket before (5.13) and Remark 4.1 that when $\boldsymbol{k}$ is fixed, $\mathcal{E}$ can be fixed, and we can assume $\mathcal{E}<\ell^{-\ell}$ for any $\ell\gg\boldsymbol{k}$. We here emphasis that the $\mathcal{E}$ used in the above design of the system of inequations in (5.20) is exactly the same as that used in the local bijectivity of Keller maps in (5.13). There is no any problem in doing this since our design does not need to use the local bijectivity of Keller maps, we only use the local bijectivity of Keller maps to show that the set V_0 we defined is nonempty [in the sense of defining the system (5.20), $\ell,\boldsymbol{k},\mathcal{E}$ are simply some chosen (and fixed) positive real numbers, cf. Remark 5.3].

Let $(p_0,p_1)\in V_0$. In (5.20) (i), if the equality occurs in the first inequality, or two equalities simultaneously occur in the second and third inequalities, then $|x_1|=|x_0|=\boldsymbol{k}$, but by (5.20) (ii), $|y_1|>\gamma_{\boldsymbol{k},\boldsymbol{k}}$, a contradiction with the definition of $\gamma_{\boldsymbol{k},\boldsymbol{k}}$.

If the equality occurs in the last inequality of (5.20) (i), then we can obtain that $|x_1|\sim\ell$ (when $\ell\gg\boldsymbol{k}$ and $\boldsymbol{k}$ is regarded as fixed; cf. Remarks 5.3 and 5.8), $|x_0|\preceq\ell^{1+\delta}$, but by (5.20) (ii),

$$|y_1|\succeq\ell^{1+\delta_1}\succ|x_1|.\tag{5.21}$$

Again by (1.14), we must have either $h_{p_0,p_1}\sim|x_0|\sim|y_0|$ or $h_{p_0,p_1}\sim|x_1|\sim|y_1|$. In any case we have $h_{p_0,p_1}\preceq\ell^{1+\delta}<\ell^{1+\frac{1}{m}}$, but then $|x_1+y_1|\ge|y_1|-|x_1|\sim|y_1|\succ h_{p_0,p_1}^{\frac{m}{m+1}}$, a contradiction with (1.13). Hence Theorem 1.5 (1) (ii) holds.

Next, we want to choose suitable u,v such that (5.20) holds for (q_0,q_1) [defined in (5.13)], i.e.,

$$\text{(i) } 1\le|1+u\mathcal{E}|^{1+\delta-\delta_1-\ell^{-3}}\le|1+s\mathcal{E}|\le|1+u\mathcal{E}|^{1+\delta}\le\ell^{1+\delta},\quad\text{(ii) }\frac{|1+v\mathcal{E}|+\mathcal{E}^3}{|1+u\mathcal{E}|^{1+\delta_1}}\ge1+\mathcal{E}^2.\tag{5.22}$$

The strict inequality automatically holds in the last inequality of (5.22) (i) (we can assume $\mathcal{E} < \ell^{-\ell}$, cf. Remark 4.1). We take

$$u = 1, \quad v = \frac{a_{\boldsymbol{k}} + 1 + \delta - \delta_1}{b_{\boldsymbol{k}}}, \qquad \text{and} \quad s = 1 + \delta - \delta_1 + O(\mathcal{E})^1, \tag{5.23}$$

where the last equation is obtained by (5.14). Then by comparing the coefficients of $\mathcal{E}^1$, one can easily see that all inequalities in (5.22) (i) are strict inequalities. Further, the coefficient of $\mathcal{E}^1$ in the left hand-side of (5.22) (ii) is $\frac{a_{\boldsymbol{k}}+1+\delta-\delta_1}{b_{\boldsymbol{k}}} - (1+\delta_1) > 0$ by (5.19). We see that $(q_0, q_1) \in V_0$, i.e., $V_0 \neq \emptyset$, a contradiction with Assumption 5.1. We have Lemma 5.7. □

Lemma 5.9. *For any fixed $\delta \in \mathbb{R}_{>0}$, we have $(1-\delta^5)b_{\boldsymbol{k}} \le 1 + a_{\boldsymbol{k}}$ for all $\boldsymbol{k} \gg 1$.*

Proof. Let $\boldsymbol{k} \gg 1$ and we assume $\mathcal{E} < \boldsymbol{k}^{-\boldsymbol{k}}$ (cf. Remark 4.1). Define V_0 to be the subset of V consisting of elements $(p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big)$ satisfying [again our purpose is to design the following to satisfy Theorem 1.5 (1); cf. Remarks 5.3 and 5.8]

$$\begin{aligned} &\text{(i) } (1-\delta^5)^{1+\boldsymbol{k}^{-3}} \le \big(\boldsymbol{k}^{-1}|x_1|\big)^{1+\boldsymbol{k}^{-3}} \le \boldsymbol{k}^{-1}|x_0| \le \big(\boldsymbol{k}^{-1}|x_1|\big)^{1-\boldsymbol{k}^{-3}} \le 1, \\ &\text{(ii) } \frac{\gamma_{\boldsymbol{k},\boldsymbol{k}}^{-1}|y_1| + \mathcal{E}^3}{\big(\boldsymbol{k}^{-1}|x_1|\big)^{1-\delta^5}} \ge 1 + \mathcal{E}^2. \end{aligned} \tag{5.24}$$

We have (1.19) and (1.21).

Let $(p_0, p_1) \in V_0$. If the equality occurs in the first inequality of (5.24) (i), then we obtain that $|x_1| = (1-\delta^5)\boldsymbol{k}$, $|x_0| \le \boldsymbol{k}$, and by (5.24) (ii), Lemma 5.6, we have

$$|y_1| > (1-\delta^5)^{1-\delta^5}\gamma_{\boldsymbol{k},\boldsymbol{k}} > \Big(1 - \delta^5 + \delta^{10} + O(\delta)^{15}\Big)\boldsymbol{k} > |x_1|. \tag{5.25}$$

As before, we would obtain that $|x_1 + y_1| \ge |y_1| - |x_1| \sim |y_1| \succeq \boldsymbol{k} \succ \boldsymbol{k}^{\frac{m}{m+1}} \sim h_{p_0,p_1}^{\frac{m}{m+1}}$ (when $\boldsymbol{k} \gg 1$, cf. Remarks 5.3 and 5.8), a contradiction with (1.13).

If two equalities simultaneously occur in the second and third inequalities of (5.24) (i), or the equality occurs in the last inequality, then $|x_1| = |x_0| = \boldsymbol{k}$, but by (5.24) (ii), $|y_1| > \gamma_{\boldsymbol{k},\boldsymbol{k}}$, a contradiction with definition (1.17). Hence Theorem 1.5 (1) (ii) holds.

Next, we want to choose suitable u, v such that (5.24) holds for (q_0, q_1) [defined in (5.13)], i.e.,

$$\begin{aligned} &\text{(i) } (1-\delta^5)^{1+\boldsymbol{k}^{-3}} \le |1 + u\mathcal{E}|^{1+\boldsymbol{k}^{-3}} \le |1 + s\mathcal{E}| \le |1 + u\mathcal{E}|^{1-\boldsymbol{k}^{-3}} \le 1, \\ &\text{(ii) } \frac{|1 + v\mathcal{E}| + \mathcal{E}^3}{|1 + u\mathcal{E}|^{1-\delta^5}} \ge 1 + \mathcal{E}^2. \end{aligned} \tag{5.26}$$

The strict inequality automatically holds in the first inequality of (5.26) (i). Take [the last equation is obtained from (5.14)]

$$u = -1, \quad v = -\frac{1 + a_{\boldsymbol{k}}}{b_{\boldsymbol{k}}}, \qquad \text{and} \quad s = -1 + O(\mathcal{E})^1. \tag{5.27}$$

By comparing the coefficients of $\mathcal{E}^1$, we see that all inequalities in (5.26) (i) are strict inequalities. Further, the coefficient of $\mathcal{E}^1$ in the left hand-side of (5.26) (ii) is $1 - \delta^5 - \frac{1+a_{\boldsymbol{k}}}{b_{\boldsymbol{k}}}$, which is positive if the assertion of the lemma is not true; in this case, we see that $(q_0, q_1) \in V_0$, i.e., $V_0 \neq \emptyset$, and we obtain a contradiction with Assumption 5.1, namely, we have Lemma 5.9. $\square$

The above two lemmas show that $a_{\boldsymbol{k}} \geq \frac{1-\delta^4(1+\delta)}{\delta^4}$. Since δ is arbitrarily sufficiently small number, we see that $a_{\boldsymbol{k}}$ (thus also $b_{\boldsymbol{k}}$) is unbounded, i.e.,

$$\lim_{\boldsymbol{k}\to\infty} a_{\boldsymbol{k}} = \lim_{\boldsymbol{k}\to\infty} b_{\boldsymbol{k}} = \infty, \qquad \text{and} \qquad \lim_{\boldsymbol{k}\to\infty} \frac{a_{\boldsymbol{k}}}{b_{\boldsymbol{k}}} = 1. \tag{5.28}$$

Remark 5.10. From the proofs above, one can see that in order to achieve our task, we must choose the power of $|x_1|$ in (5.2) (b), (5.20) (ii) and (5.24) (ii) [i.e., κ_8 in (1.19)] to be different from 1. In case $\kappa_8 > 1$ as in (5.23), we must choose v to be positive and $v > \kappa_8 u$ [so that $|y_1|$ can grow faster than $|x_1|$ as in (5.21)]; while in case $\kappa_8 < 1$ we must choose κ_8 to be independent of $\boldsymbol{k}$ as in (5.24) (ii), and choose v to be negative but bigger than $\kappa_8 u$ as in (5.27) so that (5.22) (ii) and (5.26) (ii) can hold [such choice of κ_8 can guarantee that $|y_1|$ can descend much slower than $|x_1|$ as in (5.25)]. However because of (5.28), our task becomes extremely difficult, simply because of the fact that any choice of $v > \kappa_8 u$ will force $s = -a_{\boldsymbol{k}} u + b_{\boldsymbol{k}} v + O(\mathcal{E})^1$ to be too large [which in turn will push $|x_0|$ to grow too fast], thus we have to choose v to be smaller than u, but such a choice will force $|y_1|$ to grow slower (or descend faster) than $|x_1|$ and then we are unable to obtain a contradiction if we use the previous design — this forces us to design a very complicated system in (5.31).

Finally we are able to obtain the following (which is the most difficulty part of the paper).

Lemma 5.11. *Theorem* 1.5 (1) *holds.*

Proof. We first fix some choices of positive numbers satisfying,

$$1 \ll \ell_0 := \delta_0^{-1} \ll \ell_1 := \delta_1^{-1} \ll \ell_2 := \delta_2^{-1} \ll \ell := \delta^{-1} \ll \boldsymbol{k} \ll \mathcal{E}^{-1}. \tag{5.29}$$

For instance, it is enough to take $\ell_0 = 10^{100}$, $\ell_1 = \ell_0^{\ell_0}$, $\ell_2 = \ell_1^{\ell_1}$, $\ell = \ell_2^{\ell_2}$, $\boldsymbol{k} = (\ell \boldsymbol{s}_0)^\ell$ [where $\boldsymbol{s}_0$ is given in (3.50)] and $\mathcal{E} < \boldsymbol{k}^{-\boldsymbol{k}}$. For any $(p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big) \in V$, we denote [recall from (5.9), (5.28) that $|\bar{x}_0| = |\bar{x}_1| = \boldsymbol{k}$, $|\bar{y}_1| = \gamma_{\boldsymbol{k},\boldsymbol{k}}$, and $\alpha_0 > 0$],

$$X_0 = \frac{x_0}{\bar{x}_0}, \quad X_1 = \frac{x_1}{\bar{x}_1}, \quad Y_1 = \frac{y_1}{\bar{y}_1}, \quad \widetilde{X}_0 = \big((1+\alpha_0\mathcal{E})X_0\big)^\ell, \quad \alpha_0 = \ell_0 b_{\boldsymbol{k}} - \ell_0^4 \delta > 0. \tag{5.30}$$

We now define V_0 to be the subset of V consisting of elements $(p_0, p_1) = \big((x_0, y_0), (x_1, y_1)\big)$ satisfying the following [one will see from the proof below why we have to design such a complicated system; we suggest that readers do not need to check details at this moment — we will explain everything when our arguments are carried on step by step so that all will become clear; note that throughout the rest of this section, some multi-valued functions may appear in some expressions; for instance, B_1 defined in (5.31) (v) is a multi-valued function on $\widetilde{X}_0$, X_1, Y_1; from our arguments below one can see that locally there always exists a unique choice of each multi-valued function satisfying (5.31), therefore globally there exists a unique choice of each multi-valued function; in addition, all B_1, B_2, B_3 are locally smooth functions on $\widetilde{X}_0$, X_1, Y_1],

$$\text{(i) } 1 \le |B_1|^{\frac{5\delta_0}{8}-3\delta_0^2} \le |B_2| \le |B_1|^{\frac{5\delta_0}{8}} \le \ell_1^{\frac{5\delta_0}{8}},$$

$$\text{(ii) } (1-\delta)|B_1|^{-\ell_0^2} \le |\widetilde{X}_0| \le \ell_2, \qquad \text{(iii) } (1-\delta)|B_1|^{\ell_0-3} \le |X_1| \le \ell_2,$$

$$\text{(iv) } B_3' := |B_3 B_1^{-\frac{\ell_0^2}{2}+\frac{5}{8}+\frac{3\delta_0}{8}}| + (|x_1|+|y_1|)\mathcal{E}^3 \ge 1+\mathcal{E}^2, \quad \text{(v) } B_1 = \frac{Y_1\Big(\frac{1}{2}\big(1+\frac{\delta_0^2}{2}\big)+\frac{1}{2}\big(1-\frac{\delta_0^2}{2}\big)\widetilde{X}_0^2 B_1^{2\ell_0^2}\Big)}{X_1^{1+\delta_0^2+\delta_0^3}\widetilde{X}_0 B_3^2},$$

$$\text{(vi) } B_2 = \frac{1}{2-\widetilde{X}_0 X_1^{\ell_0^2+2+2\delta_0-\delta_0^4} Y_1^{-\ell_0^2} B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}}, \qquad \text{(vii) } B_3 = \frac{X_1^{\ell_0^2}}{\widetilde{X}_0 Y_1^{\ell_0^2+1}}. \tag{5.31}$$

Then (5.31) can be rewritten as the form in (1.20). We remark that the main purpose of the initial conditions (5.31) (ii), (iii) is to guarantee that we have (5.32) (2), (7), (8), which are extremely crucial in the proof of Lemma 5.11.

Now we divide the proof of the lemma into three steps.

Step 1: We want to prove that when conditions (5.31) hold, we have the following [in particular, we have (1.21)],

$$\text{(1) } |X_0| < |\widetilde{X}_0|^{\delta} = 1+O(\delta)^1, \qquad \text{(2) } Y_1 = X_1 + O(\delta)^2, \qquad \text{(3) } B_3'' := |B_3 B_1^{-\frac{\ell_0^2}{2}+\frac{5}{8}+\frac{3\delta_0}{8}}| > 1,$$

$$\text{(4) } B_1 = \widetilde{X}_0 X_1^{2-\delta_0^2-\delta_0^3}\Big(\frac{1}{2}\big(1+\frac{\delta_0^2}{2}\big)+\frac{1}{2}\big(1-\frac{\delta_0^2}{2}\big)\widetilde{X}_0^2 B_1^{2\ell_0^2}\Big) + O(\delta)^2,$$

$$\text{(5) } B_2 = \frac{1}{2-\widetilde{X}_0 X_1^{2+2\delta_0-\delta_0^4} B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}} + O(\delta)^2, \qquad \text{(6) } B_3 = \frac{1}{\widetilde{X}_0 X_1} + O(\delta)^2,$$

$$\text{(7) } (1-\delta)|B_1|^{-\ell_0^2} < |\widetilde{X}_0| < \ell_2, \qquad \text{(8) } (1-\delta)|B_1|^{\ell_0-3} < |X_1| < \ell_2,$$

$$\text{(9) } \widetilde{X}_0 = \frac{A_1\Big(1-\sqrt{1-\big(1-\frac{\delta_0^4}{4}\big)A_2}\Big)}{1-\frac{\delta_0^2}{2}}, \quad \text{where}$$

$$\text{(10) } A_1 = Y^{-1} X_1^{\delta_0^2+\delta_0^3} B_3^2 B_1^{-2\ell_0^2+1}, \quad \text{(11) } A_2 = Y^2 X_1^{-2(\delta_0^2+\delta_0^3)} B_3^{-4} B_1^{2\ell_0^2-2}, \quad \text{(12) } Y = \frac{Y_1}{X_1}. \tag{5.32}$$

By (5.30), (5.31) (i), (ii), we obtain (5.32) (1). . To prove (5.32) (2), for the sake of convenience to state our arguments, we regard $\boldsymbol{k}$ as a variable and take $\boldsymbol{k} \gg \ell$ [which means that other elements in (5.29) are regarded as fixed and we choose $\boldsymbol{k}$ to be sufficiently larger than ℓ, cf. Remark 5.8; in this sense, $1 \sim_{\boldsymbol{k}} \ell_0 \sim_{\boldsymbol{k}} \ell_1 \sim_{\boldsymbol{k}} \ell_2 \sim_{\boldsymbol{k}} \ell \prec_{\boldsymbol{k}} \boldsymbol{k} \prec_{\boldsymbol{k}} \mathcal{E}^{-1}$; here to avoid confusion, we use the subscript "$_{\boldsymbol{k}}$" to indicate that $\boldsymbol{k}$ is regarded as a variable (similar notations will be also used below)]. Then $|B_1| \sim_{\boldsymbol{k}} |B_2| \sim_{\boldsymbol{k}} 1$ and $|X_0| \preceq_{\boldsymbol{k}} 1 \sim_{\boldsymbol{k}} |X_1|$ by (5.31) (i), (iii), (5.32) (1). By notations (5.9), (5.30), we have $|x_0| = \boldsymbol{k}|X_0| \preceq_{\boldsymbol{k}} \boldsymbol{k}|X_1| = |x_1| \sim_{\boldsymbol{k}} \boldsymbol{k}$. Thus by (1.14), we must have

$$h_{p_0,p_1} \sim_{\boldsymbol{k}} |y_1| \sim_{\boldsymbol{k}} |x_1| \sim_{\boldsymbol{k}} \boldsymbol{k}. \tag{5.33}$$

Write $x_1 = -y_1(1+\mu_1)$ for some $\mu_1 \in \mathbb{C}$, then by (1.13),

$$|y_1\mu_1| = |x_1+y_1| < h_{p_0,p_1}^{\frac{m}{m+1}} \sim_{\boldsymbol{k}} |y_1|^{\frac{m}{m+1}}, \tag{5.34}$$

i.e., $\mu_1 \preceq_{\boldsymbol{k}} |y_1|^{-\frac{1}{m+1}} \sim_{\boldsymbol{k}} \boldsymbol{k}^{-\frac{1}{m+1}} \ll \delta^2$. Thus $|\mu_1| = O(\delta)^2$. Similarly, we can write $\bar{x}_1 = -\bar{y}_1(1+\bar{\mu}_1)$ with $|\bar{\mu}_1| = O(\delta)^2$ (cf. Lemma 5.6). Hence

$$X_1 = \frac{(\bar{x}_1^{-1}x_1)Y_1}{\bar{y}_1^{-1}y_1} = \frac{(1+\mu_1)Y_1}{1+\bar{\mu}_1} = Y_1\Big(1+O(\delta)^2\Big),$$

which with (5.31) (iii) gives (5.32) (2). By notations (5.9), (5.30), and (5.31) (iii), (5.32) (4), we see that $(|x_1|+|y_1|)\mathcal{E}^3 < \mathcal{E}^2$, which with (5.31) (iv) implies (5.32) (3). By (5.31) (i)–(iii), (5.32) (2), we have

$$a^{0+O(\delta)^1} = 1+O(\delta)^1, \quad a\Big(1+O(\delta)^1\Big) = a+O(\delta)^1 \ \text{ for all } \ a\in A_0 \ \text{ or } \ a^{-1}\in A_0, \ \text{ where}$$

$$A_0 := \Big\{\widetilde{X}_0, X_1, Y_1, B_1, B_2, B_3, \Big(\frac{1}{2}\big(1+\frac{\delta_0^2}{2}\big)+\frac{1}{2}\big(1-\frac{\delta_0^2}{2}\big)\widetilde{X}_0^2B_1^{2\ell_0^2}\Big), 2-\widetilde{X}_0X_1^{2+2\delta_0-\delta_0^4}B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}\Big\}.$$

Thus we have (5.32) (4)–(6) by (5.31) (v)–(vii). To prove (5.32) (7), (8), this time we regard $\ell_2 = \delta_2^{-1}$ as a variable (then $1 \sim_{\ell_2} \ell_0 \sim_{\ell_2} \ell_1 \prec_{\ell_2} \ell_2 \prec_{\ell_2} \ell \prec_{\ell_2} \boldsymbol{k} \prec_{\ell_2} \mathcal{E}^{-1}$). We have $|B_1| \sim_{\ell_2} |B_2| \sim_{\ell_2} 1$ by (5.31) (i). Then $|\widetilde{X}_0|, |X_1|, |B_3| \succeq_{\ell_2} 1$ by the first inequalities of (5.31) (i), (ii) and (5.32) (3). Thus $|\widetilde{X}_0| \sim_{\ell_2} 1 \sim_{\ell_2,} |X_1|$ by (5.32) (6). In particular we have the last inequalities of (5.32) (7), (8).

Now assume $(1-\delta)|B_1|^{-\ell_0^2} \ge |\widetilde{X}_0|$. Then by (5.31) (ii), $|\widetilde{X}_0| = (1-\delta)|B_1|^{-\ell_0^2}$. By (5.32) (3), (6), and the fact that $|B_1| \ge 1$, we can obtain the following from (5.32) (4),

$$\begin{aligned}
1 &\le \Big|B_1^{-1}\widetilde{X}_0^{-1+\delta_0^2+\delta_0^3}B_3^{-2+\delta_0^2+\delta_0^3}\Big|\Big(\frac{1}{2}\Big(1+\frac{\delta_0^2}{2}\Big)+\frac{1}{2}\Big(1-\frac{\delta_0^2}{2}\Big)|\widetilde{X}_0B_1^{\ell_0^2}|^2\Big)+O(\delta)^2 \\
&\le (1-\delta)^{-1+\delta_0^2+\delta_0^3}|B_1|^{-1-\ell_0^2(-1+\delta_0^2+\delta_0^3)+(2-\delta_0^2-\delta_0^3)(-\frac{\ell_0^2}{2}+\frac{5}{8}+\frac{3\delta_0}{8})}\Big(1+\Big(-1+\frac{\delta_0^2}{2}\Big)\delta+O(\delta)^2\Big)+O(\delta)^2 \\
&\le \Big(1+\Big(-\frac{\delta_0^2}{2}+O(\delta_0)^3\Big)\delta+O(\delta)^2\Big)|B_1|^{-\frac{1}{4}+O(\delta_0)^1}+O(\delta)^2 \\
&\le 1+\Big(-\frac{\delta_0^2}{2}+O(\delta_0)^3\Big)\delta+O(\delta)^2 < 1,
\end{aligned} \tag{5.35}$$

a contradiction. This proves (5.32) (7). Next assume $(1-\delta)|B_1|^{\ell_0-3} \ge |X_1|$. Then $|X_1| = (1-\delta)|B_1|^{\ell_0-3}$ by (5.31) (iii). By (5.31) (i), (5.32) (3), (5), (6), we obtain

$$\begin{aligned}
2 &\le |B_2|^{-1}+|\widetilde{X}_0X_1^{2+2\delta_0-\delta_0^4}B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}|+O(\delta)^2 \le |B_1|^{-\frac{5\delta_0}{8}}+|B_3^{-1}X_1^{1+2\delta_0-\delta_0^4}B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}|+O(\delta)^2 \\
&\le 1+(1-\delta)^{1+2\delta_0-\delta_0^4}|B_1|^{-\frac{\ell_0^2}{2}+\frac{5}{8}+\frac{3\delta_0}{8}+(1+2\delta_0-\delta_0^4)(\ell_0-3)+\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}+O(\delta)^2 \\
&\le 1+\Big(1+\big(-1+O(\delta_0)^1\big)\delta+O(\delta)^2\Big)|B_1|^{-\frac{45}{8}\delta_0+O(\delta_0)^2}+O(\delta)^2 \\
&\le 2+\Big(-1+O(\delta_0)^1\Big)\delta+O(\delta)^2 < 2,
\end{aligned} \tag{5.36}$$

which is again a contradiction. This proves (5.32) (8). Finally, if we regard (5.31) (v) as an equation on $\widetilde{X}_0$, then there are two solutions for $\widetilde{X}_0$, one is stated as in (5.32) (9). We prove as follows that the other solution does not satisfy our requirement: note that locally there is only one choice of $\widetilde{X}_0$, thus globally there is only one choice of $\widetilde{X}_0$; in Step 3, we will show that we can choose $(q_0, q_1) \in V_0$ such that (5.32) (9) holds; thus it holds globally.

Step 2: Now let $(p_0,p_1)\in V_0$. First assume in (5.31) (i), the equality occurs in the first inequality, or two equalities simultaneously occur in the second and third inequalities. Then $|B_1|=|B_2|=1$ and $|B_3|>1$ by (5.32) (3). We have $|\widetilde X_0|\le 1+O(\delta)^1$, $|X_1|\ge 1+O(\delta)^1$ by (5.32) (7), (8). Thus by (5.32) (3), (6), we obtain

$$|a|=1+O(\delta)^1 \ \text{ for } \ a\in A_1:=\big\{\widetilde X_0, X_1, \widetilde Y_1:=Y_1X_1^{-(1+\delta_0^2+\delta_0^3)}, B_3\big\}. \tag{5.37}$$

By (5.31) (vi), (v), we immediately obtain

$$\text{(i) } |\widetilde X_0|<|X_1^{\ell_0^2}Y_1^{-\ell_0^2-1}|, \qquad \text{(ii) } |Y_1|^{2\ell_0^2+1}<|X_1|^{2\ell_0^2+2+2\delta_0-\delta_0^4}, \ \text{ or } \ |Y_1|<|X_1|^{1+\frac{\delta_0^2}{2}+O(\delta_0)^3}. \tag{5.38}$$

By (5.31) (v), we obtain (for convenience, we denote $\boldsymbol{x}_0=|\widetilde X_0|$, $\boldsymbol{x}_1=|X_1|$, $\boldsymbol{y}=|\widetilde Y_1|$, $\boldsymbol{b}=|B_3|$; since $\boldsymbol{b}>1$ we have the strict inequality),

$$1\le \boldsymbol{y}\boldsymbol{x}_0^{-1}\boldsymbol{b}^{-2}\Big(\frac12\big(1+\frac{\delta_0^2}{2}\big)+\frac12\big(1-\frac{\delta_0^2}{2}\big)\boldsymbol{x}_0^2\Big)<\beta_1:=\boldsymbol{y}\boldsymbol{x}_0^{-1}\Big(\frac12\big(1+\frac{\delta_0^2}{2}\big)+\frac12\big(1-\frac{\delta_0^2}{2}\big)\boldsymbol{x}_0^2\Big). \tag{5.39}$$

We claim

$$\text{(i) } \boldsymbol{x}_0<\boldsymbol{y}^{2\ell_0^2+\delta_0^5} \ \text{ if } \ \boldsymbol{y}\ge 1, \quad \text{or} \quad \text{(ii) } \boldsymbol{x}_0<\boldsymbol{y}^{2\ell_0^2-\delta_0^5} \ \text{ if } \ \boldsymbol{y}<1. \tag{5.40}$$

Say $\boldsymbol{y}\ge 1$ and $\boldsymbol{x}_0\ge \boldsymbol{y}^{2\ell_0+\delta_0^5}$. Noting that β_1 is a strictly decreasing function on $\boldsymbol{x}_0$ when other variables are fixed and when all variables satisfy (5.37) (since $\frac{\partial\beta_1}{\partial\boldsymbol{x}_0}|_{(\boldsymbol{x}_0,\boldsymbol{y})=(1,1)}=-\frac{\delta_0^2}{2}<0$ and $0<\delta\ll\delta_0$), we obtain from (5.39),

$$1<\beta_1\Big|_{\boldsymbol{x}_0=\boldsymbol{y}^{2\ell_0^2+\delta_0^5}}=\beta_2:=\boldsymbol{y}^{-2\ell_0^2+1-\delta_0^5}\Big(\frac12\big(1+\frac{\delta_0^2}{2}\big)+\frac12\big(1-\frac{\delta_0^2}{2}\big)\boldsymbol{y}^{4\ell_0^2+2\delta_0^5}\Big). \tag{5.41}$$

Noting that β_2 is a strictly decreasing on $\boldsymbol{y}$ (as $\frac{d\beta_2}{d\boldsymbol{y}}|_{\boldsymbol{y}=1}=-\frac{\delta_0^7}{2}<0$), we obtain that $\boldsymbol{y}<1$, a contradiction with the assumption. This proves (5.40) (i). Similarly, we have (5.40) (ii) if $\boldsymbol{y}<1$. This shows that in general we have $\boldsymbol{x}_0<\boldsymbol{y}^{2\ell_0^2+O(\delta_0)^5}$, i.e., $|\widetilde X_0|<|Y_1^{2\ell_0+O(\delta_0)^5}X_1^{-2\ell_0^2(1+\delta_0^2+\delta_0^3)+O(\delta_0)^5}|$. Using this in (5.31) (vi), one can easily see that $1<|Y_1^{\ell_0^2+O(\delta_0)^5}X_1^{-\ell_0^2-\delta_0^4+O(\delta_0)^5}|$, i.e., $|X_1|^{1+\delta_0^4}<|Y_1|^{1+O(\delta_0)^5}$. This with (5.38) (ii) implies that $k:=|Y_1|>1$ and $k_1:=|X_1|<k$. Further $k_0:=|X_0|<|\widetilde X_0|^\delta<1$ by (5.32) (1), (5.38) (i). By notations (5.9), (5.30), we see that $|x_1|=k_1\boldsymbol{k}<k\boldsymbol{k}$, $|x_0|=k_0\boldsymbol{k}<\boldsymbol{k}<k\boldsymbol{k}$, $|y_1|=k\gamma_{\boldsymbol{k},\boldsymbol{k}}$. We obtain [where the first inequality follows from Theorem 1.4 (iii), while the second from definition (1.17)],

$$\gamma_{k\boldsymbol{k},k\boldsymbol{k}}>\gamma_{|x_0|,|x_1|}\ge|y_1|=k\gamma_{\boldsymbol{k},\boldsymbol{k}}, \tag{5.42}$$

which is a contradiction with Lemma 5.2.

Next assume in (5.31) (i), the equality occurs in the last inequality, i.e., $|B_1|=\ell_1$. By the second and third inequalities of (5.31) (i), we have $|B_2|=\ell_1^{\frac{5\delta_0}{8}+O(\delta_0)^2}\succ_{\ell_1}1$, which with (5.32) (5) implies

$$|\widetilde X_0X_1^{2+O(\delta_0)^1}B_1^{\frac{\ell_0^2}{2}(1+O(\delta_0)^1)}|=|\widetilde X_0X_1^{2+2\delta_0-\delta_0^4}B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac38}|\sim_{\ell_1}1. \tag{5.43}$$

Note from (5.32) (4), (5.35) that $\alpha_1 := |B_1^{-1}\widetilde{X}_0 X_1^{2-\delta_0^2-\delta_0^3}| \preceq_{\ell_1} |B_1|^{-\frac{1}{4}+O(\delta_0)^1} \prec_{\ell_1} 1$. Thus (5.32) (4) shows that we must have $\alpha_2 := |\widetilde{X}_0^2 B_1^{2\ell_0^2}| \succ_{\ell_1} 1$ and

$$|\widetilde{X}_0^3 X_1^{2+O(\delta_0)^1} B_1^{2\ell_0^2}| = |\widetilde{X}_0^3 X_1^{2-\delta_0^2-\delta_0^3} B_1^{2\ell_0^2}| = \alpha_1\alpha_2 \sim_{\ell_1} 1. \tag{5.44}$$

Thus we can easily obtain from (5.43), (5.44),

$$\text{(i) } |\widetilde{X}_0| \sim_{\ell_1} |B_1|^{-\frac{3\ell_0^2}{4}(1+O(\delta_0)^1)}, \qquad \text{(ii) } |X_1| \sim_{\ell_1} |B_1|^{\frac{\ell_0^2}{8}(1+O(\delta_0)^1)}. \tag{5.45}$$

From this and (5.32) (2), (6), (11), (12), we have

$$|A_2| \sim_{\ell_1} |\widetilde{X}_0^4 X_1^{4+O(\delta_0)^1} B_1^{2\ell_0^2(1+O(\delta_0)^1)}| \sim_{\ell_1} |B_1|^{-\frac{\ell_0^2}{2}(1+O(\delta_0)^1)} \prec_{\ell_1} 1. \tag{5.46}$$

Thus we can expand $\alpha_3 := 1 - \sqrt{1-\big(1-\frac{\delta_0^4}{4}\big)A_2}$ as a power series of A_2 to obtain

$$\alpha_3 = \frac{1}{2}\Big(1-\frac{\delta_0^4}{4}\Big)A_2 + O(A_2)^1 \sim_{\ell_1} A_2. \tag{5.47}$$

Then by (5.32) (9)–(12), we have

$$|\widetilde{X}_0| \sim_{\ell_1} |A_1A_2| \sim_{\ell_1} |X_1^{0+O(\delta_0)^1} B_3^{-2} B_1^{-1}| \sim_{\ell_1} |X_1^{2+O(\delta_0)^1} \widetilde{X}_0^2 B_1^{-1}| \sim_{\ell_1} |B_1|^{-\frac{5\ell_0^2}{4}(1+O(\delta_0)^1)}, \tag{5.48}$$

which is a contradiction with (5.45) (i).

Finally by (5.32) (7), (8), we see that no equality can occur in any inequality of (5.31) (ii), (iii). This proves that Theorem 1.5 (1) (ii) holds.

Step 3: Now we want to choose suitable u, v such that (5.31) holds for (q_0, q_1). Note from notations (5.9), (5.30) that setting (p_0, p_1) to (q_0, q_1) implies that X_0, X_1, Y_1 are set to $1+s\mathcal{E}$, $1+u\mathcal{E}$, $1+v\mathcal{E}$ respectively. We take,

$$\text{(i) } u = 0, \qquad \text{(ii) } v = -\ell_0, \qquad \text{(iii) } s = s_0 + O(\mathcal{E})^1 \ \text{ with } \ s_0 = -\ell_0 b_{\boldsymbol{k}}, \tag{5.49}$$

where (5.49) (iii) is obtained from (5.14). Thus we obtain from (5.30) that $\widetilde{X}_0 = 1 + \tilde{s}\mathcal{E} + O(\mathcal{E})^2$ with $\tilde{s} = \ell(\alpha_0 + s_0) = -\ell_0^4$. Then we see from (5.31) (vii) that $B_3 = 1 + c_3\mathcal{E} + O(\mathcal{E})^2$ with $c_3 = \ell_0^2 u - \tilde{s} - (\ell_0^2+1)v = \ell_0^4 + \ell_0^3 + \ell_0$. We can unique choose B_1 of the form $B_1 = 1 + c_1\mathcal{E} + O(\mathcal{E})^2$ such that (5.31) (v) holds, where c_1 is determined from (5.31) (v) as follows,

$$c_1 = v - (1+\delta_0^2+\delta_0^3)u - \tilde{s} - 2c_3 + \frac{1}{2}\Big(1-\frac{\delta_0^2}{2}\Big)(2\tilde{s} + 2\ell_0^2 c_1), \tag{5.50}$$

and we solve that $c_1 = \frac{\ell_0^2(4+4\delta_0-\delta_0^2+6\delta_0^3)}{2-3\delta_0^2}$. Then we see from (5.31) (vi) that $B_2 = 1 + c_2\mathcal{E} + O(\mathcal{E})^2$ with

$$c_2 = \tilde{s} + (\ell_0^2 + 2 + 2\delta_0 - \delta_0^4)u - \ell_0^2 v + \Big(\frac{\ell_0^2}{2} - \ell_0 + \frac{3}{8}\Big)c_1 = \frac{5\ell_0}{4} - \frac{51}{16} + O(\delta_0)^1. \tag{5.51}$$

Then one can easily observe that both sides of (5.32) (9) are elements of the form $1+O(\mathcal{E})^1$ [thus (5.32) (9) holds for (q_0,q_1)]. Now we obtain (note that $c_1 = 2\ell_0^2+2\ell_0+\frac{5}{2}+O(\delta_0)^1$),

$$\begin{aligned} 0 \ &< \ \Big(\frac{5\delta_0}{8}-3\delta_0^2\Big)c_1 = \frac{5\ell_0}{4}-\frac{19}{4}+O(\delta_0)^1 \ < \ c_2 \\ &< \ \frac{5\delta_0 c_1}{8} = \frac{5\ell_0}{4}+\frac{5}{4}+O(\delta_0)^1. \end{aligned} \tag{5.52}$$

We see that all inequalities in (5.31) (i) are strict inequalities. Obviously, all inequalities in (5.31) (ii), (iii) are strict inequalities. Further, the coefficient of $\mathcal{E}^1$ in B_3' is

$$c_3' = c_3 + \Big(-\frac{\ell_0^2}{2}+\frac{5}{8}+\frac{3\delta_0}{8}\Big)c_1 = \frac{7}{16}+O(\delta_0)^1 > 0, \tag{5.53}$$

i.e., the inequality in (5.31) (iv) is a strict inequality. Hence $(q_0,q_1)\in V_0$, i.e., $V_0\neq\emptyset$, and we obtain a contradiction with Assumption 5.1. The above shows that Assumption 5.1 must be wrong, namely, we have the lemma. □

6. Proof of Theorem 1.5 (2)

Proof of Theorem 1.5 (2). Now we prove Theorem 1.5 (2). Let $(p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)\in V_0$, i.e., (1.19) or (1.20) [cf. (5.31)] holds. Note that (1.19)–(1.21) imply that $x_0,x_1,y_1\neq 0$. Similar to (4.5) and (5.10), we define

$$F_0 = F\big(x_0(1+x), y_0+y\big), \quad F_1 = F\big(x_1(1+x), y_1(1+y)\big), \tag{6.1}$$

and define G_0, G_1 similarly. Define q_0,q_1 accordingly [similar to (4.8) and (5.13)],

$$q_0 := (\dot{x}_0,\dot{y}_0) = \big(x_0(1+s\mathcal{E}), y_0+t\mathcal{E}\big), \quad q_1 := (\dot{x}_1,\dot{y}_1) = \big(x_1(1+u\mathcal{E}), y_1(1+v\mathcal{E})\big). \tag{6.2}$$

As in (4.10) and (5.14), we have,

$$s = s_0 + O(\mathcal{E})^1, \quad s_0 = au+bv. \tag{6.3}$$

Remark 6.1. We remark that the $\mathcal{E}$ here shall be regarded to be different from that in the previous results, here $\mathcal{E}$ may be much smaller than the previous $\mathcal{E}$. If we denote the previous $\mathcal{E}$ as $\mathcal{E}'$, whenever necessary we can assume our new $\mathcal{E}$ satisfies that $\mathcal{E} < \mathcal{E}'^{\mathcal{E}'^{-1}}$.

Now we consider two cases.

Case 1: *First assume we have the case* (1.19). If no equality occurs in any inequality of (1.19) (a), then we only need to consider (1.22), which can be easily done. Thus by Theorem 1.5 (1) (ii), we may assume that the equality occurs in the third inequality of (1.19) (a) [the proof is similar if the equality occurs in the second inequality of (1.19) (a)], i.e.,

$$\kappa_4' - 1 = 0, \quad \text{where} \quad \kappa_4' = \frac{\kappa_4|x_1|^{\kappa_5}}{\kappa_3|x_0|}. \tag{6.4}$$

Then we only need to choose (q_0,q_1) to satisfy the third inequality of (1.20) (a) and (1.22). By writing [using (6.3)]

$$\frac{\kappa_4|\dot{x}_1|^{\kappa_5}}{\kappa_3|\dot{x}_0|} = \big|(1+u\mathcal{E})^{\kappa_5}(1+s\mathcal{E})^{-1}\big| = \big|1+\big((\kappa_5-a)u-bv\big)\mathcal{E}+\cdots\big|, \tag{6.5}$$

we need to choose u, v so that (q_0, q_1) satisfies the following, for some $\tilde{\alpha}_i \in \mathbb{C}$,

$$\text{(i) } C_1 := \big|1 + \big((\kappa_5 - a)u - bv\big)\mathcal{E} + (\tilde{\alpha}_3 u^2 + \tilde{\alpha}_4 uv + \tilde{\alpha}_5 v^2)\mathcal{E}^2\big| - 1 + O(\mathcal{E})^3 \ge 0,$$

$$\text{(ii) } C_2 := |1 + v\mathcal{E}| + \kappa_7' - (1 + \kappa_7')|1 + \kappa_8 u\mathcal{E} + \tilde{\alpha}_6 u^2\mathcal{E}^2| + O(\mathcal{E})^3 > 0, \tag{6.6}$$

where $\kappa_7' = \kappa_7 |y_1|^{-1}$, and (6.6) (ii) is obtained by rewriting (1.22) as $\frac{|\dot{y}_1| + \kappa_7}{|y_1|} - \frac{|y_1| + \kappa_7}{|y_1|} \cdot \frac{|\dot{x}_1|^{\kappa_8}}{|x_1|^{\kappa_8}} > 0$.

First assume $b_1 := \kappa_5 - a \ne 0$. Then by setting

$$u = b b_1^{-1} v + (\tilde{\alpha}_7 v^2 + \tilde{\alpha}_8 w)\mathcal{E}, \tag{6.7}$$

for some $\tilde{\alpha}_i, w \in \mathbb{C}$ with $w_{\rm re} > 0$ [cf. Convention 2.1 (1) (2) for notations "$_{\rm re}$", "$_{\rm im}$"] so that C_1 can become [one can easily observe that when we substitute u in (6.6) (i) by (6.7), there are always solutions of $\tilde{\alpha}_7, \tilde{\alpha}_8$ so that C_1 can become the following form]

$$C_1 = |1 + w\mathcal{E}^2| - 1 + O(\mathcal{E})^3 = w_{\rm re}\mathcal{E}^2 + O(\mathcal{E})^3 > 0, \tag{6.8}$$

i.e., (6.6) (i) holds. Using (6.7) in (6.3), we obtain, for some $\tilde{\alpha}_i \in \mathbb{C}$,

$$s = \tilde{\alpha}_0 v + (\tilde{\alpha}_1 v^2 + \tilde{\alpha}_2 w)\mathcal{E} + O(\mathcal{E})^2 \ . \tag{6.9}$$

Using (6.7) and (6.9) in (6.6) (ii), we can then rewrite C_2 as

$$C_2 = |1 + v\mathcal{E}| + \kappa_7' - (1 + \kappa_7')\big|1 + \tilde{\alpha}_9 v\mathcal{E} + (\tilde{\alpha}_{10} v^2 + \tilde{\alpha}_{11} w)\mathcal{E}^2\big| + O(\mathcal{E})^3 > 0, \tag{6.10}$$

for some $\tilde{\alpha}_i \in \mathbb{C}$. By comparing the coefficients of $\mathcal{E}^1$ in (6.10), we immediately obtain that if $c_0 := 1 - (1 + \kappa_7')\tilde{\alpha}_9 \ne 0$, then we can always choose v [with $(c_0 v)_{\rm re} > 0$] to satisfy (6.10).

Assume $c_0 = 0$ (then $\tilde{\alpha}_9$ is real). Then we see that C_2 in (6.10) is an $O(\mathcal{E})^2$ element. In this case, since we do not know what are values of $\tilde{\alpha}_{10}, \tilde{\alpha}_{11}$, our strategy is to compute the following coefficient [cf. Convention 2.1 (2) for notation $\mathrm{C_{oeff}}$; also note that we use $v_{\rm re}^2$ to denote $(v_{\rm re})^2$],

$$\tilde{\beta} = \tilde{\beta}_1 + \tilde{\beta}_2 \ \text{ with } \ \tilde{\beta}_1 = \mathrm{C_{oeff}}(C_2, v_{\rm re}^2\mathcal{E}^2) \text{ and } \tilde{\beta}_2 = \mathrm{C_{oeff}}(C_2, v_{\rm im}^2\mathcal{E}^2). \tag{6.11}$$

We observe the important fact that $\tilde{\alpha}_{10}$ does not contribute to $\tilde{\beta}$ by noting the following

$$(\tilde{\alpha}_{10} v^2\mathcal{E}^2)_{\rm re} = \big(\tilde{\alpha}_{10\,{\rm re}}(v_{\rm re}^2 - v_{\rm im}^2) + 2\tilde{\alpha}_{10\,{\rm im}} v_{\rm re} v_{\rm im}\big)\mathcal{E}^2, \tag{6.12}$$

and that the imaginary part of $\tilde{\alpha}_{10} v^2\mathcal{E}^2$ can only contribute an $O(\mathcal{E})^4$ element to C_2 in (6.10). Thus for the purpose of computing $\tilde{\beta}$, we may assume $\tilde{\alpha}_{10} = \tilde{\alpha}_{11} = 0$ (then the computation becomes much easier). Since $\tilde{\alpha}_9$ is real, it is straightforward to compute that

$$\tilde{\beta} = \frac{\kappa_7'}{2(\kappa_7' + 1)} > 0, \tag{6.13}$$

by (6.10) [remark: the fact that $\kappa_7 > 0$, i.e., $\tilde{\beta} > 0$ is very crucial for the inequation (6.10) being solvable for any unknown $\tilde{\alpha}_i \in \mathbb{C}$ in (6.10), cf. Remark 6.2]. By (6.11) and (6.13), either $\tilde{\beta}_1 > 0$ or $\tilde{\beta}_2 > 0$, and we can then choose v with $v_{\rm re}^2$ being sufficiently larger than $v_{\rm im}^2$ or respectively with $v_{\rm im}^2$ being sufficiently larger than $v_{\rm re}^2$, to guarantee that (6.6) (ii) [i.e., (6.10)] holds (when w is fixed). This proves Theorem 1.5 (2) for the case that $b_1 \ne 0$.

Assume $b_1 = \kappa_5 - a = 0$. We simply set $v = 0$. Then we have the following.

- If C_1 in (6.6) (i) is independent of u, then $C_1 = 0$, i.e., (6.6) (i) holds automatically, in this case we can simply choose u with $u_{\rm re} > 0$ so that $C_2 = (1+\kappa'_7)\kappa_8 u_{\rm re}\mathcal{E} + O(\mathcal{E})^2 > 0$, i.e., (6.6) (ii) holds.
- Otherwise, $C_1 = |1 + b' u^k \mathcal{E}^k| - 1 + O(\mathcal{E})^{k+1}$ for some $b' \in \mathbb{C}_{\neq 0}$ and $k \in \mathbb{Z}_{\geq 2}$, and we can always choose $u \in \mathbb{C}$ with $(b'u^k)_{\rm re} > 0$ and $u_{\rm re} > 0$ (such u always exists simply because $k \geq 2$) to guarantee that both of (6.6) hold.

This completes the proof of Theorem 1.5 (2) for the case (1.19).

Remark 6.2. (cf. Remark 1.6) Assume that we have the following inequation on variable u, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}_{>0}$, and $a_1, a_2, a_3 \in \mathbb{C}$ are some unknown complex numbers:

$$\alpha_1|1 + a_1 u\mathcal{E} + a_2 u^2\mathcal{E}^2 + a_3\mathcal{E}^2|^{\beta_1} < \alpha_2|1 + u\mathcal{E}|^{\beta_2} + \alpha_1 - \alpha_2 + O(\mathcal{E})^3. \tag{6.14}$$

Then from the proof of (6.6), one can see that this inequation is solvable for **any unknown** $a_1, a_2, a_3 \in \mathbb{C}$ if and only if $\alpha_1 > \alpha_2$.

Case 2: *Now assume we have the case* (1.20). First we remark that we have designed the last two terms in (1.20) (c) [cf. (5.31) (iii)] in order that we can solve the inequations below. By Theorem 1.5 (1) (ii), no matter whether the equality occurs in the second or third inequality of (1.20) (a), the two inequations we need to consider can be always stated as the following, for some $\tilde{\alpha}_i \in \mathbb{C}$, $\kappa'_0 \in \mathbb{R}_{>0}$,

$$\text{(i) } C'_1 := \big|1 + (\tilde{\alpha}_1 u + \tilde{\alpha}_2 v)\mathcal{E}\big| - 1 + O(\mathcal{E})^2 \geq 0,$$

$$\text{(ii) } C'_2 := \kappa'_0\big|1 + (\tilde{\alpha}_3 u + \tilde{\alpha}_4 v)\mathcal{E}\big| + |1 + u\mathcal{E}| + |1 + v\mathcal{E}| - (\kappa'_0 + 2) + O(\mathcal{E})^2 > 0. \tag{6.15}$$

First assume $\tilde{\alpha}_1 \neq 0$. Then as in Case 1, we can take $u = -\tilde{\alpha}_1^{-1}\tilde{\alpha}_2 v + (\beta_1 v^2 + \beta_2 w)\mathcal{E}$ for some $\beta_i, w \in \mathbb{C}$ with $w_{\rm re} > 0$ so that C'_1 has the form as in (6.8) [i.e., (6.15) (i) holds], and (6.15) (ii) becomes the following [cf. (6.10)], for some $\tilde{\alpha}_i \in \mathbb{C}$,

$$\begin{aligned} C'_2 = \kappa'_0\big|1 + \tilde{\alpha}_5 v\mathcal{E} + (\tilde{\alpha}_6 v^2 + \tilde{\alpha}_7 w)\mathcal{E}^2\big| + \big|1 + \tilde{\alpha}_8 v\mathcal{E} + (\tilde{\alpha}_9 v^2 + \tilde{\alpha}_{10} w)\mathcal{E}^2\big| \\ + |1 + v\mathcal{E}| - (\kappa'_0 + 2) + O(\mathcal{E})^3 > 0. \end{aligned} \tag{6.16}$$

As in (6.10), we see that if $c_0 := \kappa'_0\tilde{\alpha}_5 + \tilde{\alpha}_8 + 1 \neq 0$, we can always choose $v \in \mathbb{C}$ with $(c_0 v)_{\rm re} > 0$ to satisfy (6.16). Thus assume $c_0 = 0$. Then as in (6.13), one can compute

$$\tilde{\beta} := \mathrm{C_{oeff}}(C'_2, v_{\rm re}^2\mathcal{E}^2) + \mathrm{C_{oeff}}(C'_2, v_{\rm im}^2\mathcal{E}^2) = \frac{1}{2}\Big((\kappa'_0\tilde{\alpha}_{5\,\rm re} + 1)^2 + \kappa'_0(\kappa'_0 + 1)\tilde{\alpha}_{5\,\rm im}^2 + \kappa'_0\tilde{\alpha}_{5\,\rm re}^2 + 1\Big) > 0. \tag{6.17}$$

Thus we can choose v with $v_{\rm re}^2$ being sufficiently larger than $v_{\rm im}^2$ if $\mathrm{C_{oeff}}(C'_2, v_{\rm re}^2\mathcal{E}^2) > 0$ or with $v_{\rm im}^2$ being sufficiently larger than $v_{\rm re}^2$ if $\mathrm{C_{oeff}}(C'_2, v_{\rm im}^2\mathcal{E}^2) > 0$, to guarantee that (6.16) holds (when w is fixed). This proves Theorem 1.5 (2) for the case that $\tilde{\alpha}_1 \neq 0$.

Now assume $\tilde{\alpha}_1 = 0$. By symmetry, we may also assume $\tilde{\alpha}_2 = 0$. Then we have one of the following,

$$\text{(i) } C'_1 = 0, \qquad \text{or} \qquad \text{(ii) } C'_1 = |1 + g(u, v)\mathcal{E}^k| - 1 + O(\mathcal{E})^{k+1}, \tag{6.18}$$

for some nonzero homogeneous polynomial $g(u, v)$ of u, v of degree $k \in \mathbb{Z}_{\geq 2}$ [assume we have (6.18) (ii) as (6.15) (i) holds trivially for case (6.18) (i)]. In case $c_1 := \kappa'_0\tilde{\alpha}_3 + 1 \neq 0$ [see (6.15) (ii)], we can solve the problem as follows: First take $v = \alpha u$ for some $\alpha \in \mathbb{C}$ with $g(u, \alpha u) \neq 0$ [in this case $g(u, \alpha u) = b' u^k$ for some $b' \in \mathbb{C}_{\neq 0}$] and with $|\alpha|$ being sufficiently small, then we choose $u \in \mathbb{C}$ with $(c_1 u)_{\rm re} > 0$ so that (6.15) (ii) holds [since $|\alpha|$ is sufficiently small (say we choose α with $0 < |\alpha| \ll |c_1|$), our choice of u with $(c_1 u)_{\rm re} > 0$ can guarantee that (6.15) (ii) holds], and further $(b' u^k)_{\rm re} > 0$ (this can be done since $k \geq 2$). If $c_2 := \kappa_0' \tilde\alpha_4 + 1 \neq 0$, we can solve the problem symmetrically.

Assume $c_1 = c_2 = 0$ [then C_2' becomes an $O(\mathcal{E})^2$ element]. One can easily compute as in Case 1,

$$\mathrm{C}_{\rm oeff}(C_2', u_{\rm re}^2 \mathcal{E}^2) + \mathrm{C}_{\rm oeff}(C_2', u_{\rm im}^2) = \mathrm{C}_{\rm oeff}(C_2', v_{\rm re}^2 \mathcal{E}^2) + \mathrm{C}_{\rm oeff}(C_2', v_{\rm im}^2) = \frac{\kappa_0' + 1}{2\kappa_0'} > 0. \qquad (6.19)$$

If $g(u, v)$ does not depend on v [i.e., $g(u, v) = b' u^k$ for some $b' \in \mathbb{C}_{\neq 0}$], then we can first choose $u \in \mathbb{C}$ to satisfy that $g(u, v)_{\rm re} > 0$ then choose $v \in \mathbb{C}$ with $v_{\rm re}^2$ being sufficiently larger than $v_{\rm im}^2$ if $\mathrm{C}_{\rm oeff}(C_2', v_{\rm re}^2 \mathcal{E}^2) > 0$ or with $v_{\rm im}^2$ being sufficiently larger than $v_{\rm re}^2$ if $\mathrm{C}_{\rm oeff}(C_2', v_{\rm im}^2 \mathcal{E}^2) > 0$, to guarantee that $\mathrm{C}_{\rm oeff}(C_2', \mathcal{E}^2) > 0$, i.e., (6.15) (ii) holds. Thus assume $g(u, v)$ depends on v. We set $v = \alpha u$ with $\alpha, u \in \mathbb{C}$ being determined later such that $|\alpha|$ is sufficiently small. Then (6.18) (ii) and (6.15) (ii) become the following, for some $\tilde\alpha_{11} \in \mathbb{C}$, and some non-constant polynomial $g_0(\alpha)$ of α [where the term $-\kappa_0'^{-1}(1 + \alpha) u \mathcal{E}$ in C_2' is obtained by the assumption that $c_1 = c_2 = 0$, i.e., $\tilde\alpha_3 = -\kappa_0'^{-1} = \tilde\alpha_4$],

(i) $C_1' = |1 + g_0(\alpha) u^k \mathcal{E}^k| - 1 + O(\mathcal{E})^{k+1} \geq 0,$

(ii) $C_2' = \kappa_0' |1 - \kappa_0'^{-1}(1 + \alpha) u \mathcal{E} + \tilde\alpha_{11} u^2 \mathcal{E}| + |1 - u\mathcal{E}| + |1 + \alpha u \mathcal{E}| - (\kappa_0' + 2) + O(\mathcal{E})^3 > 0.$ (6.20)

One can compute the following,

$$\beta := \mathrm{C}_{\rm oeff}(C_2', u_{\rm re}^2 \mathcal{E}^2) + \mathrm{C}_{\rm oeff}(C_2', u_{\rm im}^2) = \frac{1}{2\kappa_0'} \Big((\alpha_{\rm re} + 1)^2 + \alpha_{\rm im}^2 + \kappa_0' + \alpha_{\rm im}^2 \kappa_0' + \alpha_{\rm re}^2 \kappa_0' \Big) > 0. \qquad (6.21)$$

We can always choose $u \in \mathbb{C}$ with $u_{\rm re}^2$ being sufficiently larger than $u_{\rm im}^2$ if $\mathrm{C}_{\rm oeff}(C_2', u_{\rm re}^2 \mathcal{E}^2) > 0$ or with $u_{\rm im}^2$ being sufficiently larger than $u_{\rm re}^2$ if $\mathrm{C}_{\rm oeff}(C_2', u_{\rm im}^2 \mathcal{E}^2) > 0$, to guarantee that $\mathrm{C}_{\rm oeff}(C_2', \mathcal{E}^2) > 0$, i.e., (6.20) (ii) holds; and further $(g_0(\alpha) u^k)_{\rm re} > 0$ by some suitable choose of $\alpha \in \mathbb{C}$ [when $|\alpha|$ is sufficiently small, one can guarantee that the choice of α does not affect the inequality in (6.20) (ii) by noting that when $|\alpha|$ is sufficiently small, β defined in (6.21) is bigger than a positive number which is independent of α], i.e., (6.20) (i) holds.

This proves Theorem 1.5. □

7. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.2. To prove Theorem 1.2 (i), we use Theorem 1.5. Denote [cf. (1.19), (1.20)]

$$L = \{\ell_{p_0, p_1} \,|\, (p_0, p_1) \in V_0\}, \quad \ell = \sup L \in \mathbb{R}_{>0} \cup \{+\infty\} \text{ (the supremum of } L). \qquad (7.1)$$

By definition, there exists a sequence $(p_{0i}, p_{1i}) := \big((x_{0i}, y_{0i}), (x_{1i}, y_{1i})\big) \in V_0$, $i = 1, 2, ...$, i.e., $p_{0i} \neq p_{1i}$ and [assume we have case (1.19) as the proof for the case (1.20) is exactly similar],

$$\sigma(p_{0i}) = \sigma(p_{1i}), \;\; \kappa_0 \leq \kappa_1 |x_{1i}|^{\kappa_2} \leq \kappa_3 |x_{0i}| \leq \kappa_4 |x_{1i}|^{\kappa_5} \leq \kappa_6, \;\; \ell_i := \frac{|y_{1i}| + \kappa_7}{|x_{1i}|^{\kappa_8}} \geq \kappa_9, \qquad (7.2)$$

such that $\lim_{i\to\infty}\ell_i=\ell$ (cf. Remark 1.6). By (1.21), $|x_{0i}|$, $|x_{1i}|$ are bounded. By (1.13), we see that $|y_{0i}|,|y_{1i}|$ are also bounded [as in the proof of Theorem 1.4 (i)]. Thus $\ell\in\mathbb{R}_{>0}$. By replacing the sequence by a subsequence, we may assume

$$\lim_{i\to\infty}(p_{0i},p_{1i})=(p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)\in\mathbb{C}^4. \tag{7.3}$$

First suppose $p_0=p_1$. Then by (7.3), for any neighborhood $\mathcal{O}_{p_0}$ of p_0, there exists N_0 such that $p_{0i},p_{1i}\in\mathcal{O}_{p_0}$ when $i>N_0$, but $p_{0i}\ne p_{1i}$, $\sigma(p_{0i})=\sigma(p_{1i})$, which is a contradiction with the local bijectivity of Keller maps. Thus $p_0\ne p_1$. By taking the limit $i\to\infty$ in (7.2), we see that (1.21) is satisfied by x_0,x_1,y_1 and all conditions in (1.19) hold for (p_0,p_1) [in case (1.20), one can see from (5.31)–(5.32) that all functions f_i's on x_0,x_1,y_1 are still well-defined]. Thus $(p_0,p_1)\in V_0$. Therefore by Theorem 1.5 (2), there exists $(q_0,q_1)=\big((\dot{x}_0,\dot{y}_0),(\dot{x}_1,\dot{y}_1)\big)\in V_0$ such that $\ell_{q_0,q_1}>\ell_{p_0,p_1}=\ell$, a contradiction with (7.1). This proves that (1.15) is not true, i.e., we have Theorem 1.2 (i).

To prove Theorem 1.2 (ii), as in (4.4) and (5.10), take $(p_0,p_1)=\big((x_0,y_0),(x_1,y_1)\big)\in V$ and set (and define G_0,G_1 similarly)

$$F_0=F(x_0+\alpha_0x,y_0+y),\qquad F_1=F(x_1+\alpha_1x,y_1+y),\quad\text{where} \tag{7.4}$$

$$\alpha_0=\begin{cases}1 & \text{if } x_0=\xi_0,\\ x_0-\xi_0 & \text{else,}\end{cases}\qquad \alpha_1=\begin{cases}1 & \text{if } x_1=\xi_1,\\ x_1-\xi_1 & \text{else.}\end{cases} \tag{7.5}$$

Define q_0,q_1 accordingly [cf. (4.8) and (5.13)]. Then we have as in (4.10) and (6.3) [here we use symbols a,b,c,d instead of $\tilde a,\tilde b,\tilde c,\tilde d$ in (4.7), (4.10)],

$$s=au+bv+O(\mathcal{E})^1. \tag{7.6}$$

Note from Theorem 1.2 (i) that $(x_0,x_1)\ne(\xi_0,\xi_1)$.

First suppose $x_0\ne\xi_0$, $x_1\ne\xi_1$ (then $\alpha_0=x_0-\xi_0$, $\alpha_1=x_1-\xi_1$). In this case, we need to choose u,v such that,

$$C_0:=\beta_0|1+s\mathcal{E}|^2+\beta_1|1+u\mathcal{E}|^2-(\beta_0+\beta_1)<0, \tag{7.7}$$

where $\beta_0=|x_0-\xi_0|^2,\ \beta_1=|x_1-\xi_1|^2$. Using (7.6) in (7.7), we immediately see (by comparing the coefficients of $\mathcal{E}^1$) that if $b\ne0$ or $a\ne-\beta_0\beta_1^{-1}$, then we have a solution for (7.7). Thus assume $b=0$, $a=-\beta_1\beta_0^{-1}$ [then $d\ne0$ in (4.7) and a is real]. In this case, using arguments after (4.11), we have the similar versions of either (4.21) and (4.22), or else (4.25) and (4.26), i.e.,

$$u=\hat u\mathcal{E}^{k-1},\qquad v=d^{-1}w-d^{-1}c\hat u\mathcal{E}^{k-1},\qquad s=(a\hat u+b'w^k)\mathcal{E}^{k-1}+O(\mathcal{E}^k),\quad\text{or else} \tag{7.8}$$

$$u=u_1\boldsymbol{i}\mathcal{E}^{i_0-1},\quad v=d^{-1}w-d^{-1}cu_1\boldsymbol{i}\mathcal{E}^{i_0-1},\quad s\mathcal{E}=au_1\boldsymbol{i}\mathcal{E}^{i_0}+b''u_1\boldsymbol{i}w^{i_0}\mathcal{E}^{2i_0}+O(\mathcal{E}^{2i_0+1}), \tag{7.9}$$

for some $b',b'',u,w\in\mathbb{C}_{\ne0}$, $u_1\in\mathbb{R}_{\ne0}$, $k,i_0\in\mathbb{Z}_{>0}$, one can again find a solution for the inequation (7.7).

Now if $x_0=\xi_0$ (thus $x_1\ne\xi_1$), then the first term of C_0 becomes $|s\mathcal{E}|^2=O(\mathcal{E})^2$ and we can easily choose any u with $u_{\rm re}<0$ to satisfy that $C_0<0$. Similarly, if $x_1=\xi_1$ (thus $x_0\ne\xi_0$), then the second term of C_0 becomes $|u\mathcal{E}|^2=O(\mathcal{E})^2$ and we can easily choose u with $(au)_{\rm re}<0$ (in case $a\ne0$) or v with $(bv)_{\rm re}<0$ (in case $b\ne0$) to satisfy that $C_0<0$. This proves Theorem 1.2. $\square$

Proof of Theorem 1.1. Finally we are able to prove Theorem 1.1. The second assertion of Theorem 1.1 follows from [8, 24]. To prove the first statement, assume conversely that there exists a Jacobian pair $(F,G)\in\mathbb{C}[x,y]^2$ satisfying (1.5) such that (1.3) holds. Then we have Theorem 1.2. Similar to the proof of Theorem 1.2, denote $\boldsymbol{D}=\{\boldsymbol{d}_{p_0,p_1}\,|\,(p_0,p_1)\in V\}$ [cf. (1.9)], and set $\boldsymbol{d}=\inf\boldsymbol{D}\in\mathbb{R}_{\ge0}$ (the infimum of $\boldsymbol{D}$). By definition, there exists a sequence $(p_{0i},p_{1i}):=\big((x_{0i},y_{0i}),(x_{1i},y_{1i})\big)\in V$, $i=1,2,...$, such that $\lim_{i\to\infty}\boldsymbol{d}_{p_{0i},p_{1i}}=\boldsymbol{d}$. Then $\{x_{0i},x_{1i}\,|\,i=1,2,...\}$ is bounded by (1.9). Thus $\{y_{0i},y_{1i}\,|\,i=1,2,...\}$ is also bounded by (1.13). By replacing the sequence by a subsequence, we can then assume (7.3). Now arguments after (7.3) show that $(p_0,p_1)\in V$, but $(x_0,x_1)\ne(\xi_0,\xi_1)$ by Theorem 1.2 (i), i.e., $\boldsymbol{d}>0$. Then by Theorem 1.2 (ii), we can then obtain a contradiction with the definition of $\boldsymbol{d}$. This proves Theorem 1.1. □

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